The Blind Humanoid - Kinematic and Inertial Data Fusion

***Abstract*—Not done yet.**

I. Goal

The goal of this document is to provide a first draft for kinematic/inertial state estimation for humanoid robots. It builds up on the IROS2013 paper "Robust State Estimation for Legged Robots"[1]. If unfamiliar with notation or the handling of 3D rotations, please look up the corresponding sections.

II. Filter Derivation

A. State and Continuous Prediction Equations

The same robot-centric states as in [1] are being used:

$$\boldsymbol{x} := \left(\boldsymbol{r}, \boldsymbol{v}, \boldsymbol{q}, \boldsymbol{c}, \boldsymbol{d}\right) \tag{1}$$

$$:= \left({}_B\boldsymbol{r}_{BW}, -{}_B\boldsymbol{v}_B, \boldsymbol{q}_{WB}, {}_B\boldsymbol{b}_f, {}_B\boldsymbol{b}_\omega\right). \tag{2}$$

The advantage of the robot-centric choice of states is that we thereby partition the state into non-observable states (absolute position and yaw) and observable states and thus avoid numerical problems related to non-observable states. The Drawback is that the noise of the gyroscope propagates onto the position state as well. This will most probably leed to a slightly less accurate position estimate. The rest of the states should not be significantly affected.

In addition, the state vector is augmented by the full pose of the feet that are currently in contact with the ground:

$$p_i = {}_B p_{BF_i}, \tag{3}$$

$$z_i = q_{F_i B}. \tag{4}$$

F_i is the foot fixed coordinate frame and it is assumed that it is stationary when it is on the ground.

The prediction equations can be derived along similar lines as in [1]. For this we first write down the continuous prediction equations of the system which can be derived by evaluating the different kinematic relations:

$$\dot{\boldsymbol{r}} = -\left(\tilde{\boldsymbol{\omega}} - \boldsymbol{d} - \boldsymbol{w}_\omega\right)^\times \boldsymbol{r} + \boldsymbol{v}, \tag{5}$$

$$\dot{\boldsymbol{v}} = -\left(\tilde{\boldsymbol{\omega}} - \boldsymbol{d} - \boldsymbol{w}_\omega\right)^\times \boldsymbol{v} - \tilde{\boldsymbol{f}} + \boldsymbol{c} + \boldsymbol{w}_f - \boldsymbol{C}^T(\boldsymbol{q})\boldsymbol{g}, \tag{6}$$

$$\dot{\boldsymbol{q}} = \boldsymbol{C}(\boldsymbol{q})(\tilde{\boldsymbol{\omega}} - \boldsymbol{d} - \boldsymbol{w}_\omega), \tag{7}$$

$$\dot{\boldsymbol{c}} = \boldsymbol{w}_c, \tag{8}$$

$$\dot{\boldsymbol{d}} = \boldsymbol{w}_d. \tag{9}$$

$$\dot{\boldsymbol{p}}_i = -\left(\tilde{\boldsymbol{\omega}} - \boldsymbol{d} - \boldsymbol{w}_\omega\right)^\times \boldsymbol{p}_i + \boldsymbol{v} + \boldsymbol{w}_p, \tag{10}$$

$$\dot{\boldsymbol{z}}_i = \boldsymbol{C}(\boldsymbol{z}_i)(\tilde{\boldsymbol{\omega}} - \boldsymbol{d} - \boldsymbol{w}_\omega) + \boldsymbol{w}_z. \tag{11}$$

The prediction of the contact foot states are similar to the one of the base position and attitude states. We model it to be affected by some additional Gaussian white noise, w_p and w_z, in order to allow for a small amount of slippage.

B. Discretization of Prediction Equations

In contrast to [1], a simple Euler forward integration is used for discretizating the prediction equations. Still, note that for the rotational states, the step forward can be taken on the sigma algebra and then be mapped back onto SO(3):

$$\boldsymbol{q}(t_k) = \boldsymbol{q}(t_{k-1}) \boxplus \left(\Delta t_k \dot{\boldsymbol{q}}(t_{k-1})\right) \tag{12}$$

with

$$\Delta t_k = \ t_k - t_{k-1}. \tag{13}$$

This leads to:

$$\boldsymbol{r}_k = \left(I - \Delta t_k \left(\tilde{\boldsymbol{\omega}}_k - \boldsymbol{d}_{k-1} - \boldsymbol{w}_{\omega,k}\right)^\times\right)\boldsymbol{r}_{k-1} + \Delta t_k \boldsymbol{v}_{k-1} \tag{14}$$

$$\boldsymbol{v}_k = \left(I - \Delta t_k \left(\tilde{\boldsymbol{\omega}}_k - \boldsymbol{d}_{k-1} - \boldsymbol{w}_{\omega,k}\right)^\times\right)\boldsymbol{v}_{k-1} + \Delta t_k\left(\tilde{\boldsymbol{f}}_k - \boldsymbol{c}_{k-1} - \boldsymbol{w}_{f,k} + \boldsymbol{C}^T(\boldsymbol{q}_{k-1})\boldsymbol{g}\right) \tag{15}$$

$$\boldsymbol{q}_k = \boldsymbol{q}_{k-1} \otimes \exp\left(\Delta t_k(\tilde{\boldsymbol{\omega}}_k - \boldsymbol{d}_{k-1} - \boldsymbol{w}_{\omega,k})\right), \tag{16}$$

$$\boldsymbol{c}_k = \boldsymbol{c}_{k-1} + \Delta t_k \boldsymbol{w}_{c,k}, \tag{17}$$

$$\boldsymbol{d}_k = \boldsymbol{d}_{k-1} + \Delta t_k \boldsymbol{w}_{d,k}, \tag{18}$$

$$\boldsymbol{p}_{i,k} = \left(I - \Delta t_k \left(\tilde{\boldsymbol{\omega}}_k - \boldsymbol{d}_{k-1} - \boldsymbol{w}_{\omega,k}\right)^\times\right)\boldsymbol{p}_{i,k-1} + \Delta t_k\left(\boldsymbol{v}_{k-1} + \boldsymbol{w}_{p,k}\right), \tag{19}$$

$$\boldsymbol{z}_{i,k} = \exp\left(\Delta t_k \boldsymbol{w}_{z,k}\right) \otimes \boldsymbol{z}_{i,k-1} \otimes \exp\left(\Delta t_k\left(\tilde{\boldsymbol{\omega}}_k - \boldsymbol{d}_{k-1} - \boldsymbol{w}_{\omega,k}\right)\right). \tag{20}$$

C. Measurements and Update Equations

The choice of robot-centric filter states leads to almost trivial measurement equations for the forward kinematics (depending on the kinematics measurements $\tilde{\alpha}_k$):

$$p_i(\tilde{\boldsymbol{\alpha}}_k) = p_{i,k} + \boldsymbol{n}_{p,k}, \tag{21}$$

$$z_i(\tilde{\boldsymbol{\alpha}}_k) = z_{i,k} \boxplus \boldsymbol{n}_{z,k}, \tag{22}$$

D. Foot Accelerometers

Optionally the measurement from the foot acceleremoter can also be included into the state estimation. For feet that are currently in contact and thus assumed to be stationary the following measurement equation can directly be stated:

$$\tilde{\boldsymbol{f}}_{i,k} = -\boldsymbol{C}(\boldsymbol{z}_{i,k})\boldsymbol{C}^T(\boldsymbol{q}_k)\boldsymbol{g} + \boldsymbol{c}_i + \boldsymbol{n}_{f,i,k}. \tag{23}$$

c_i is the bias term that affects the accelerometer of foot i and needs to be integrated into the filter state as well. The corresponding prediction equation is analogous to the one of the accelerometer bias term of the main body.

E. Observability Analysis

For the case without foot accelerometers the observability is as follows: For general motions only absolute position and yaw angle are not observable. If $(\tilde{\omega} - d) = 0$ the rank deficiency of the observability matrix increase by 2. For the special case $(\tilde{\omega} - d) \perp C^T(q)g$ the rank deficiency increase by 1. The (simplified) observability matrix leading to those statements looks as follows:

$$\boldsymbol{O}(\boldsymbol{x}, \boldsymbol{u}) = \begin{bmatrix} 0 & \boldsymbol{I} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\boldsymbol{C}^T(\boldsymbol{q})\boldsymbol{g}^\times & \boldsymbol{I} & 0 & 0 & 0 \\ 0 & 0 & (\tilde{\boldsymbol{\omega}} - \boldsymbol{d})^\times \boldsymbol{C}^T(\boldsymbol{q})\boldsymbol{g}^\times & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boldsymbol{I} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boldsymbol{I} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boldsymbol{I} \end{bmatrix}$$

F. Choice of Filter and Jacobians

Various kinds of filters can be used together with the above equations in order to perform state estimation. The easiest way is probably to use an UKF because it does not require the computation of any Jacobians and can easily handle correlated noise terms (as is the case if one uses a ”velocity filter”). If one still wants to implement an EKF the Jacobians will have to be computed. For the prediction step this gives:

$$\boldsymbol{G}_{k-1} = \begin{bmatrix} 0 & -\Delta t_k \boldsymbol{r}_{k-1}^{\times} & 0 & 0 & 0 & 0 \\ -\Delta t_k \boldsymbol{I} & -\Delta t_k \boldsymbol{v}_{k-1}^{\times} & 0 & 0 & 0 & 0 \\ 0 & -\boldsymbol{C}(\boldsymbol{q}_{k-1})\boldsymbol{\Gamma}_{\boldsymbol{1}}(\Delta t_k(\tilde{\boldsymbol{\omega}}_k - \boldsymbol{d}_{k-1}) & 0 & 0 & 0 & 0 \\ 0 & 0 & \Delta t_k \boldsymbol{I} & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta t_k \boldsymbol{I} & 0 & 0 \\ 0 & -\Delta t_k \boldsymbol{p}_{k-1}^{\times} & 0 & 0 & \Delta t_k \boldsymbol{I} & 0 \\ 0 & -\boldsymbol{C}(\boldsymbol{z}_{k-1})\boldsymbol{\Gamma}_{\boldsymbol{1}}(\Delta t_k(\tilde{\boldsymbol{\omega}}_k - \boldsymbol{d}_{k-1}) & 0 & 0 & 0 & \Delta t_k \boldsymbol{I} \end{bmatrix}$$

For the regular measurement update this gives:

$$\boldsymbol{H}_k = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \boldsymbol{I} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boldsymbol{I} \end{bmatrix}$$

And if you add the foot accelerometer measurements one gets (this needs to be completed with the identity matrix for the entry corresponding to the bias of the foot accelerometer i):

$$\boldsymbol{H}_{i,k} = \begin{bmatrix} 0 & 0 & -\boldsymbol{C}(\boldsymbol{z}_{i,k})\boldsymbol{C}^T(\boldsymbol{q}_k)\boldsymbol{g}^{\times} & 0 & 0 & 0 & \left(\boldsymbol{C}(\boldsymbol{z}_{i,k})\boldsymbol{C}^T(\boldsymbol{q}_k)\boldsymbol{g}\right)^{\times} \end{bmatrix}$$

G. Velocity Filter

Instead of augmenting the state by the pose of the feet, one can also directly include the ”no motion” assumption of the feet into the filter. An advantage of this approach is that one can easily implement slip detection and rejection. Furthermore on a velocity level the results will be as least as good as if using the pose measurements. A drawback is that you get correlation betwenn prediction step and update step and thus need to implement a special filter (this is very easy if using an UKF).

The foot accelerometers measurements can still be easily included during stance phase (the measurement will get crosscorrelated to the forward kinematics).

H. Observability Analysis Extended

This section compares the observability characteristics of the pointfoot constraint versus the one of the flat foot constraint. This part looks at the non-robot-centric states. We have the following dynamics (without noise and we use $r = \boldsymbol{r}$, $v = \boldsymbol{v}$, $q = \boldsymbol{q}$, $C = \boldsymbol{C}(\boldsymbol{q})$, $c = \boldsymbol{c}$, $d = \boldsymbol{d}$, $p = \boldsymbol{p}$, $f = -(\tilde{\boldsymbol{f}} - \boldsymbol{c})$, $w = -(\tilde{\boldsymbol{\omega}} - \boldsymbol{d})$):

$$\dot{r} = v, \tag{24}$$
$$\dot{v} = -C^T f + g, \tag{25}$$
$$\dot{q} = w, \tag{26}$$
$$\dot{c} = 0, \tag{27}$$
$$\dot{d} = 0, \tag{28}$$
$$\dot{p} = 0, \tag{29}$$
$$\dot{z} = 0. \tag{30}$$

1) Pointfoot Constraint: Using the special derivatives we can evaluate the Lie derivatives and the corresponding gradients:

$$\mathcal{L}_f^0 h(x) = C(p - r) \tag{31}$$
$$\nabla \mathcal{L}_f^0 h(x) = \begin{bmatrix} -C & 0 & -(C(p-r))^\times & 0 & 0 & C & 0 \end{bmatrix} \tag{32}$$
$$\mathcal{L}_f^1 h(x) = -Cv + w^\times C(p - r) \tag{33}$$
$$\nabla \mathcal{L}_f^1 h(x) = \begin{bmatrix} -w^\times C & -C & (Cv)^\times - w^\times (C(p-r))^\times & 0 & -(C(p-r))^\times & w^\times C & 0 \end{bmatrix} \tag{34}$$
$$\mathcal{L}_f^2 h(x) = -2w^\times Cv + f - Cg + w^{\times^2} C(p - r) \tag{35}$$
$$\nabla \mathcal{L}_f^2 h(x) = \begin{bmatrix} -w^{\times^2} C & -2w^\times C & w^\times \left(2(Cv)^\times - w^\times (C(p-r))^\times\right) + (Cg)^\times & I & \frac{\partial \mathcal{L}_f^2 h(x)}{\partial d} & w^{\times^2} C & 0 \end{bmatrix} \tag{36}$$
$$\mathcal{L}_f^3 h(x) = -3w^{\times^2} Cv + 2w^\times f - 3w^\times Cg + w^{\times^3} C(p - r) \tag{37}$$
$$\nabla \mathcal{L}_f^3 h(x) = \begin{bmatrix} -w^{\times^3} C & -3w^{\times^2} C & w^\times \left(3w^\times (Cv)^\times - w^{\times^2} (C(p-r))^\times + 3(Cg)^\times\right) & 2w^\times & \frac{\partial \mathcal{L}_f^3 h(x)}{\partial d} & w^{\times^3} C & 0 \end{bmatrix} \tag{38}$$
$$\mathcal{L}_f^4 h(x) = -4w^{\times^3} Cv + 3w^{\times^2} f - 6w^{\times^2} Cg + w^{\times^4} C(p - r) \tag{39}$$
$$\nabla \mathcal{L}_f^4 h(x) = \begin{bmatrix} -w^{\times^4} C & -4w^{\times^3} C & w^{\times^2} \left(4w^\times (Cv)^\times - w^{\times^2} (C(p-r))^\times + 6(Cg)^\times\right) & 3w^{\times^2} & \frac{\partial \mathcal{L}_f^4 h(x)}{\partial d} & w^{\times^4} C & 0 \end{bmatrix} \tag{40}$$
$$\mathcal{L}_f^5 h(x) = -5w^{\times^4} Cv + 4w^{\times^3} f - 10w^{\times^3} Cg + w^{\times^5} C(p - r) \tag{41}$$
$$\nabla \mathcal{L}_f^5 h(x) = \begin{bmatrix} -w^{\times^5} C & -5w^{\times^4} C & w^{\times^3} \left(5w^\times (Cv)^\times - w^{\times^2} (C(p-r))^\times + 10(Cg)^\times\right) & 4w^{\times^3} & \frac{\partial \mathcal{L}_f^5 h(x)}{\partial d} & w^{\times^5} C & 0 \end{bmatrix} \tag{42}$$
$$\mathcal{L}_f^k h(x) = -kw^{\times^{k-1}} Cv + (k-1)w^{\times^{k-2}} f - \frac{k(k-1)}{2} w^{\times^{k-2}} Cg + w^{\times^k} C(p - r) \tag{43}$$
$$\nabla \mathcal{L}_f^k h(x) = \begin{bmatrix} -w^{\times^k} C & -kw^{\times^{k-1}} C & kw^{\times^{k-1}} (Cv)^\times - w^{\times^k} (C(p-r))^\times + \frac{k(k-1)}{2} w^{\times^{k-2}} (Cg)^\times & (k-1)w^{\times^{k-2}} & \frac{\partial \mathcal{L}_f^k h(x)}{\partial d} & w^{\times^k} C & 0 \end{bmatrix} \tag{44}$$

For continuing we will make use of the follwoing identities:

$$\left(w^{\times^k} u\right)^\times = w^\times \left(w^{\times^{k-1}} u\right)^\times - \left(w^{\times^{k-1}} u\right)^\times w^\times \tag{45}$$
$$(u)^\times = u^\times \tag{46}$$
$$\left(w^\times u\right)^\times = w^\times u^\times - u^\times w^\times \tag{47}$$
$$\left(w^{\times^2} u\right)^\times = w^{\times^2} u^\times - 2w^\times u^\times w^\times + u^\times w^{\times^2} \tag{48}$$
$$\left(w^{\times^3} u\right)^\times = w^{\times^3} u^\times - 3w^{\times^2} u^\times w^\times + 3w^\times u^\times w^{\times^2} - u^\times w^{\times^3} \tag{49}$$
$$\left(w^{\times^4} u\right)^\times = w^{\times^4} u^\times - 4w^{\times^3} u^\times w^\times + 6w^{\times^2} u^\times w^{\times^2} - 4w^\times u^\times w^{\times^3} + u^\times w^{\times^4} \tag{50}$$
$$\left(w^{\times^k} u\right)^\times = \sum_{i=0}^{k} (-1)^i \binom{k}{i} w^{\times^{k-i}} u^\times w^{\times^i} \tag{51}$$

$$\frac{\partial w^{\times^k} u}{\partial \omega} = w^\times \frac{\partial w^{\times^{k-1}} u}{\partial \omega} - \left(w^{\times^{k-1}} u\right)^\times \tag{52}$$
$$\frac{\partial w^{\times^k} u}{\partial \omega} = -\sum_{i=0}^{k-1} w^{\times^{k-i-1}} \left(w^{\times^i} u\right)^\times \tag{53}$$
$$\frac{\partial w^\times u}{\partial \omega} = -u^\times \tag{54}$$
$$\frac{\partial w^{\times^2} u}{\partial \omega} = -w^\times u^\times - (w^\times u)^\times \tag{55}$$
$$\frac{\partial w^{\times^3} u}{\partial \omega} = -w^{\times^2} u^\times - w^\times (w^\times u)^\times - (w^{\times^2} u)^\times \tag{56}$$
$$\frac{\partial w^{\times^4} u}{\partial \omega} = -w^{\times^3} u^\times - w^{\times^2} (w^\times u)^\times - w^\times (w^{\times^2} u)^\times - (w^{\times^3} u)^\times \tag{57}$$

$$
\begin{aligned}
\forall k \geq 1, l \geq 2: \quad w^{\times^k}\left(w^{\times^l}u\right)^\times &= w^{\times^{k+1}}\left(w^{\times^{l-1}}u\right)^\times - w^{\times^k}\left(w^{\times^{l-1}}u\right)^\times w^\times \\
&= w^{\times^{k+1}}\left(w^{\times^{l-1}}u\right)^\times - w^{\times^{k+1}}\left(w^{\times^{l-2}}u\right)^\times w^\times + w^{\times^k}\left(w^{\times^{l-1}}u\right)^\times w^{\times^2} \\
&= w^{\times^{k+1}}\left(w^{\times^{l-1}}u\right)^\times
\end{aligned} \tag{58}
$$

$$
\forall k \geq 1, l \geq 1, m \geq 1: \quad w^{\times^k}\left(w^{\times^l}u\right)^\times w^{\times^m} = 0 \tag{59}
$$

In matrix forms the observability matrix looks as follows:

$$
\mathcal{O}(x) = \begin{bmatrix}
-C & 0 & -(C(p-r))^\times & 0 & 0 & C & 0 \\
-w^\times C & -C & (Cv)^\times - w^\times(C(p-r))^\times & 0 & -(C(p-r))^\times & w^\times C & 0 \\
-w^{\times^2}C & -2w^\times C & 2w^\times(Cv)^\times - w^{\times^2}(C(p-r))^\times + (Cg)^\times & I & 2(Cv)^\times - w^\times(C(p-r))^\times - (w^\times C(p-r))^\times & w^{\times^2}C & 0 \\
-w^{\times^3}C & -3w^{\times^2}C & 3w^{\times^2}(Cv)^\times - w^{\times^3}(C(p-r))^\times + 3w^\times(Cg)^\times & 2w^\times & A_3 & w^{\times^3}C & 0 \\
-w^{\times^4}C & -4w^{\times^3}C & 4w^{\times^3}(Cv)^\times - w^{\times^4}(C(p-r))^\times + 6w^{\times^2}(Cg)^\times & 3w^{\times^2} & A_4 & w^{\times^4}C & 0 \\
-w^{\times^5}C & -5w^{\times^4}C & 5w^{\times^4}(Cv)^\times - w^{\times^5}(C(p-r))^\times + 10w^{\times^3}(Cg)^\times & 4w^{\times^3} & A_5 & w^{\times^5}C & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix} \tag{60}
$$

with

$$
\mathcal{L}_f^k h(x) = w^{\times^k}C(p-r) - kw^{\times^{k-1}}Cv + (k-1)w^{\times^{k-2}}f - \frac{k(k-1)}{2}w^{\times^{k-2}}Cg \tag{61}
$$

$$
\begin{aligned}
A_k &:= \frac{\partial \mathcal{L}_f^k h(x)}{\partial d} \quad (\forall k \geq 3) \\
&= -\sum_{i=0}^{k-1} w^{\times^{k-i-1}}\left(w^{\times^i}C(p-r)\right)^\times + k\sum_{i=0}^{k-2} w^{\times^{k-i-2}}\left(w^{\times^i}Cv\right)^\times - (k-1)\sum_{i=0}^{k-3} w^{\times^{k-i-3}}\left(w^{\times^i}f\right)^\times + \frac{k(k-1)}{2}\sum_{i=0}^{k-3} w^{\times^{k-i-3}}\left(w^{\times^i}Cg\right)^\times
\end{aligned} \tag{62}
$$

After the first reduction we obtain (row k minus omega times row k-1, start with row 2):

$$
\mathcal{O}(x) = \begin{bmatrix}
-C & 0 & -(C(p-r))^\times & 0 & 0 & C & 0 \\
0 & -C & (Cv)^\times & 0 & -(C(p-r))^\times & 0 & 0 \\
0 & -w^\times C & w^\times(Cv)^\times + (Cg)^\times & I & 2(Cv)^\times - (w^\times C(p-r))^\times & 0 & 0 \\
0 & -w^{\times^2}C & w^{\times^2}(Cv)^\times + 2w^\times(Cg)^\times & w^\times & A_3 - w^\times\left(2(Cv)^\times - w^\times(C(p-r))^\times - (w^\times C(p-r))^\times\right) & 0 & 0 \\
0 & -w^{\times^3}C & w^{\times^3}(Cv)^\times + 3w^{\times^2}(Cg)^\times & w^{\times^2} & B_4 & 0 & 0 \\
0 & -w^{\times^4}C & w^{\times^4}(Cv)^\times + 4w^{\times^3}(Cg)^\times & w^{\times^3} & B_5 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix} \tag{63}
$$

with

$$
\begin{aligned}
B_k &:= A_k - w^\times A_{k-1} \quad (\forall k \geq 4) \\
&= -\sum_{i=0}^{k-1} w^{\times^{k-i-1}}\left(w^{\times^i}C(p-r)\right)^\times + k\sum_{i=0}^{k-2} w^{\times^{k-i-2}}\left(w^{\times^i}Cv\right)^\times - (k-1)\sum_{i=0}^{k-3} w^{\times^{k-i-3}}\left(w^{\times^i}f\right)^\times + \frac{k(k-1)}{2}\sum_{i=0}^{k-3} w^{\times^{k-i-3}}\left(w^{\times^i}Cg\right)^\times \\
&\quad + \sum_{i=0}^{k-2} w^{\times^{k-i-1}}\left(w^{\times^i}C(p-r)\right)^\times - (k-1)\sum_{i=0}^{k-3} w^{\times^{k-i-2}}\left(w^{\times^i}Cv\right)^\times + (k-2)\sum_{i=0}^{k-4} w^{\times^{k-i-3}}\left(w^{\times^i}f\right)^\times - \frac{(k-1)(k-2)}{2}\sum_{i=0}^{k-4} w^{\times^{k-i-3}}\left(w^{\times^i}Cg\right)^\times \\
&= -\left(w^{\times^{k-1}}C(p-r)\right)^\times + k\left(w^{\times^{k-2}}Cv\right)^\times - (k-1)\left(w^{\times^{k-3}}f\right)^\times + \frac{k(k-1)}{2}\left(w^{\times^{k-3}}Cg\right)^\times \\
&\quad + \sum_{i=0}^{k-3} w^{\times^{k-i-2}}\left(w^{\times^i}Cv\right)^\times - \sum_{i=0}^{k-4} w^{\times^{k-i-3}}\left(w^{\times^i}f\right)^\times + (k-1)\sum_{i=0}^{k-4} w^{\times^{k-i-3}}\left(w^{\times^i}Cg\right)^\times
\end{aligned} \tag{64}
$$

After the second reduction we obtain (row k minus omega times row k-1, start with row 3):

$$
\mathcal{O}(x) = \begin{bmatrix}
-C & 0 & -(C(p-r))^\times & 0 & 0 & C & 0 \\
0 & -C & (Cv)^\times & 0 & -(C(p-r))^\times & 0 & 0 \\
0 & 0 & (Cg)^\times & I & 2(Cv)^\times + (C(p-r))^\times w^\times & 0 & 0 \\
0 & 0 & w^\times(Cg)^\times & 0 & A_3 - w^\times\left(4(Cv)^\times - w^\times(C(p-r))^\times - 2(w^\times C(p-r))^\times\right) & 0 & 0 \\
0 & 0 & w^{\times^2}(Cg)^\times & 0 & B_4 - w^\times\left(A_3 - w^\times\left(2(Cv)^\times - w^\times(C(p-r))^\times - (w^\times C(p-r))^\times\right)\right) & 0 & 0 \\
0 & 0 & w^{\times^3}(Cg)^\times & 0 & C_5 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix} \tag{65}
$$

with

$$
\begin{aligned}
C_k &:= B_k - w^{\times} B_{k-1} \qquad (\forall k \geq 5) \\
&= -\left(w^{\times^{k-1}} C(p-r)\right)^{\times} + k\left(w^{\times^{k-2}} Cv\right)^{\times} - (k-1)\left(w^{\times^{k-3}} f\right)^{\times} + \frac{k(k-1)}{2}\left(w^{\times^{k-3}} Cg\right)^{\times} \\
&\quad + \sum_{i=0}^{k-3} w^{\times^{k-i-2}}\left(w^{\times^{i}} Cv\right)^{\times} - \sum_{i=0}^{k-4} w^{\times^{k-i-3}}\left(w^{\times^{i}} f\right)^{\times} + (k-1)\sum_{i=0}^{k-4} w^{\times^{k-i-3}}\left(w^{\times^{i}} Cg\right)^{\times} \\
&\quad + w^{\times}\left(w^{\times^{k-2}} C(p-r)\right)^{\times} - (k-1)w^{\times}\left(w^{\times^{k-3}} Cv\right)^{\times} + (k-2)w^{\times}\left(w^{\times^{k-4}} f\right)^{\times} - \frac{(k-1)(k-2)}{2} w^{\times}\left(w^{\times^{k-4}} Cg\right)^{\times} \\
&\quad - \sum_{i=0}^{k-4} w^{\times^{k-i-2}}\left(w^{\times^{i}} Cv\right)^{\times} + \sum_{i=0}^{k-5} w^{\times^{k-i-3}}\left(w^{\times^{i}} f\right)^{\times} - (k-2)\sum_{i=0}^{k-5} w^{\times^{k-i-3}}\left(w^{\times^{i}} Cg\right)^{\times} \\
&= -\left(w^{\times^{k-1}} C(p-r)\right)^{\times} + k\left(w^{\times^{k-2}} Cv\right)^{\times} - (k-1)\left(w^{\times^{k-3}} f\right)^{\times} + \frac{k(k-1)}{2}\left(w^{\times^{k-3}} Cg\right)^{\times} \\
&\quad + w^{\times}\left(w^{\times^{k-2}} C(p-r)\right)^{\times} - (k-2)w^{\times}\left(w^{\times^{k-3}} Cv\right)^{\times} + (k-3)w^{\times}\left(w^{\times^{k-4}} f\right)^{\times} - \frac{(k-1)(k-4)}{2} w^{\times}\left(w^{\times^{k-4}} Cg\right)^{\times} \\
&\quad + \sum_{i=0}^{k-5} w^{\times^{k-i-3}}\left(w^{\times^{i}} Cg\right)^{\times}
\end{aligned} \tag{66}
$$

After the third reduction we obtain (row k minus omega times row k-1, start with row 5):

$$
\mathcal{O}(x) = \begin{bmatrix}
-C & 0 & -(C(p-r))^{\times} & 0 & 0 & C & 0 \\
0 & -C & (Cv)^{\times} & 0 & -(C(p-r))^{\times} & 0 & 0 \\
0 & 0 & (Cg)^{\times} & I & 2(Cv)^{\times} + (C(p-r))^{\times} w^{\times} & 0 & 0 \\
0 & 0 & w^{\times}(Cg)^{\times} & 0 & A_3 - w^{\times}\left(4(Cv)^{\times} - w^{\times}(C(p-r))^{\times} - 2(w^{\times}C(p-r))^{\times}\right) & 0 & 0 \\
0 & 0 & 0 & 0 & B_4 - w^{\times}\left(2A_3 - w^{\times}\left(6(Cv)^{\times} - 2w^{\times}(C(p-r))^{\times} - 3(w^{\times}C(p-r))^{\times}\right)\right) & 0 & 0 \\
0 & 0 & 0 & 0 & C_5 - w^{\times}\left(B_4 - w^{\times}\left(A_3 - w^{\times}\left(2(Cv)^{\times} - w^{\times}(C(p-r))^{\times} - (w^{\times}C(p-r))^{\times}\right)\right)\right) & 0 & 0 \\
0 & 0 & 0 & 0 & D_6 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix} \tag{67}
$$

with

$$
\begin{aligned}
D_k &:= C_k - w^{\times} C_{k-1} \qquad (\forall k \geq 6) \\
&= -\left(w^{\times^{k-1}} C(p-r)\right)^{\times} + k\left(w^{\times^{k-2}} Cv\right)^{\times} - (k-1)\left(w^{\times^{k-3}} f\right)^{\times} + \frac{k(k-1)}{2}\left(w^{\times^{k-3}} Cg\right)^{\times} \\
&\quad + w^{\times}\left(w^{\times^{k-2}} C(p-r)\right)^{\times} - (k-2)w^{\times}\left(w^{\times^{k-3}} Cv\right)^{\times} + (k-3)w^{\times}\left(w^{\times^{k-4}} f\right)^{\times} - \frac{(k-1)(k-4)}{2} w^{\times}\left(w^{\times^{k-4}} Cg\right)^{\times} \\
&\quad + \sum_{i=0}^{k-5} w^{\times^{k-i-3}}\left(w^{\times^{i}} Cg\right)^{\times} \\
&\quad + w^{\times}\left(w^{\times^{k-2}} C(p-r)\right)^{\times} - (k-1)w^{\times}\left(w^{\times^{k-3}} Cv\right)^{\times} + (k-2)w^{\times}\left(w^{\times^{k-4}} f\right)^{\times} - \frac{(k-1)(k-2)}{2} w^{\times}\left(w^{\times^{k-4}} Cg\right)^{\times} \\
&\quad - w^{\times^{2}}\left(w^{\times^{k-3}} C(p-r)\right)^{\times} + (k-3)w^{\times^{2}}\left(w^{\times^{k-4}} Cv\right)^{\times} - (k-4)w^{\times^{2}}\left(w^{\times^{k-5}} f\right)^{\times} + \frac{(k-2)(k-5)}{2} w^{\times^{2}}\left(w^{\times^{k-5}} Cg\right)^{\times} \\
&\quad - \sum_{i=0}^{k-6} w^{\times^{k-i-3}}\left(w^{\times^{i}} Cg\right)^{\times} \\
&= -\left(w^{\times^{k-1}} C(p-r)\right)^{\times} + k\left(w^{\times^{k-2}} Cv\right)^{\times} - (k-1)\left(w^{\times^{k-3}} f\right)^{\times} + \frac{k(k-1)}{2}\left(w^{\times^{k-3}} Cg\right)^{\times} \\
&\quad + 2w^{\times}\left(w^{\times^{k-2}} C(p-r)\right)^{\times} - (2k-3)w^{\times}\left(w^{\times^{k-3}} Cv\right)^{\times} + (2k-5)w^{\times}\left(w^{\times^{k-4}} f\right)^{\times} - \frac{(k-1)(2k-6)}{2} w^{\times}\left(w^{\times^{k-4}} Cg\right)^{\times} \\
&\quad - w^{\times^{2}}\left(w^{\times^{k-3}} C(p-r)\right)^{\times} + (k-3)w^{\times^{2}}\left(w^{\times^{k-4}} Cv\right)^{\times} - (k-4)w^{\times^{2}}\left(w^{\times^{k-5}} f\right)^{\times} + \frac{(k-3)(k-4)}{2} w^{\times^{2}}\left(w^{\times^{k-5}} Cg\right)^{\times} \\
&= -\left(w^{\times^{k-1}} C(p-r)\right)^{\times} + k\left(w^{\times^{k-2}} Cv\right)^{\times} - (k-1)\left(w^{\times^{k-3}} f\right)^{\times} + \frac{k(k-1)}{2}\left(w^{\times^{k-3}} Cg\right)^{\times} \\
&\quad + w^{\times}\left(w^{\times^{k-2}} C(p-r)\right)^{\times} - kw^{\times}\left(w^{\times^{k-3}} Cv\right)^{\times} + (k-1)w^{\times}\left(w^{\times^{k-4}} f\right)^{\times} - \frac{(k+2)(k-3)}{2} w^{\times}\left(w^{\times^{k-4}} Cg\right)^{\times} \\
&= \left(w^{\times^{k-2}} C(p-r)\right)^{\times} w^{\times} - k\left(w^{\times^{k-3}} Cv\right)^{\times} w^{\times} + (k-1)\left(w^{\times^{k-4}} f\right)^{\times} w^{\times} + 3w^{\times}\left(w^{\times^{k-4}} Cg\right)^{\times} - \frac{k(k-1)}{2}\left(w^{\times^{k-4}} Cg\right)^{\times} w^{\times}
\end{aligned} \tag{68}
$$

After the this we can make a further reduction:

$$\mathcal{O}(x)=\begin{bmatrix} -C & 0 & -(C(p-r))^\times & 0 & 0 & C & 0\\ 0 & -C & (Cv)^\times & 0 & -(C(p-r))^\times & 0 & 0\\ 0 & 0 & (Cg)^\times & I & 2(Cv)^\times+(C(p-r))^\times w^\times & 0 & 0\\ 0 & 0 & w^\times(Cg)^\times & 0 & A_3-w^\times\left(4(Cv)^\times-w^\times(C(p-r))^\times-2(w^\times C(p-r))^\times\right) & 0 & 0\\ 0 & 0 & 0 & 0 & B_4-w^\times\left(2A_3-w^\times\left(6(Cv)^\times-2w^\times(C(p-r))^\times-3(w^\times C(p-r))^\times\right)\right) & 0 & 0\\ 0 & 0 & 0 & 0 & C_5-w^\times\left(B_4-w^\times\left(A_3-w^\times\left(2(Cv)^\times-w^\times(C(p-r))^\times-(w^\times C(p-r))^\times\right)\right)\right) & 0 & 0\\ 0 & 0 & 0 & 0 & D_6 & 0 & 0\\ 0 & 0 & 0 & 0 & D_7 & 0 & 0\\ 0 & 0 & 0 & 0 & E_8 & 0 & 0\\ 0 & 0 & 0 & 0 & E_9 & 0 & 0\\ 0 & 0 & 0 & 0 & F_{10} & 0 & 0\\ 0 & 0 & 0 & 0 & F_{11} & 0 & 0 \end{bmatrix} \tag{69}$$

with

$$\begin{aligned} E_k &:= D_k+\|w\|^2 D_{k-2} \quad (\forall k\geq 8) \\ &= -\left(w^{\times^{k-1}}C(p-r)\right)^\times + k\left(w^{\times^{k-2}}Cv\right)^\times-(k-1)\left(w^{\times^{k-3}}f\right)^\times+\frac{k(k-1)}{2}\left(w^{\times^{k-3}}Cg\right)^\times \\ &\quad + w^\times\left(w^{\times^{k-2}}C(p-r)\right)^\times-kw^\times\left(w^{\times^{k-3}}Cv\right)^\times+(k-1)w^\times\left(w^{\times^{k-4}}f\right)^\times-\frac{(k+2)(k-3)}{2}w^\times\left(w^{\times^{k-4}}Cg\right)^\times \\ &\quad +\left(w^{\times^{k-1}}C(p-r)\right)^\times-(k-2)\left(w^{\times^{k-2}}Cv\right)^\times+(k-3)\left(w^{\times^{k-3}}f\right)^\times-\frac{(k-2)(k-3)}{2}\left(w^{\times^{k-3}}Cg\right)^\times \\ &\quad -w^\times\left(w^{\times^{k-2}}C(p-r)\right)^\times+(k-2)w^\times\left(w^{\times^{k-3}}Cv\right)^\times-(k-3)w^\times\left(w^{\times^{k-4}}f\right)^\times+\frac{k(k-5)}{2}w^\times\left(w^{\times^{k-4}}Cg\right)^\times \\ &= 2\left(w^{\times^{k-2}}Cv\right)^\times-2\left(w^{\times^{k-3}}f\right)^\times+(2k-3)\left(w^{\times^{k-3}}Cg\right)^\times \\ &\quad -2w^\times\left(w^{\times^{k-3}}Cv\right)^\times+2w^\times\left(w^{\times^{k-4}}f\right)^\times-(2k-3)w^\times\left(w^{\times^{k-4}}Cg\right)^\times \\ &= -2\left(w^{\times^{k-3}}Cv\right)^\times w^\times+2\left(w^{\times^{k-4}}f\right)^\times w^\times-(2k-3)\left(w^{\times^{k-4}}Cg\right)^\times w^\times \end{aligned} \tag{70}$$

$$\begin{aligned} F_k &:= E_k+\|w\|^2 E_{k-2} \quad (\forall k\geq 10) \\ &= 2\left(w^{\times^{k-2}}Cv\right)^\times-2\left(w^{\times^{k-3}}f\right)^\times+(2k-3)\left(w^{\times^{k-3}}Cg\right)^\times \\ &\quad -2w^\times\left(w^{\times^{k-3}}Cv\right)^\times+2w^\times\left(w^{\times^{k-4}}f\right)^\times-(2k-3)w^\times\left(w^{\times^{k-4}}Cg\right)^\times \\ &\quad -2\left(w^{\times^{k-2}}Cv\right)^\times+2\left(w^{\times^{k-3}}f\right)^\times-(2k-7)\left(w^{\times^{k-3}}Cg\right)^\times \\ &\quad +2w^\times\left(w^{\times^{k-3}}Cv\right)^\times-2w^\times\left(w^{\times^{k-4}}f\right)^\times+(2k-7)w^\times\left(w^{\times^{k-4}}Cg\right)^\times \\ &= 4\left(w^{\times^{k-3}}Cg\right)^\times-4w^\times\left(w^{\times^{k-4}}Cg\right)^\times \\ &= -4\left(w^{\times^{k-4}}Cg\right)^\times w^\times \end{aligned} \tag{71}$$

This makes our matrix finite and enables the evaluation of its rank. Evaluating the different quantities yields:

$$\begin{aligned} A_3-&w^\times\left(4(Cv)^\times-w^\times(C(p-r))^\times-2(w^\times C(p-r))^\times\right) \\ &= -w^{\times^2}(C(p-r))^\times-w^\times\left(w^\times C(p-r)\right)^\times-\left(w^{\times^2}C(p-r)\right)^\times+3w^\times(Cv)^\times+3\left(w^\times Cv\right)^\times-2f^\times+3(Cg)^\times \\ &\quad -4w^\times(Cv)^\times+w^{\times^2}(C(p-r))^\times+2w^\times(w^\times C(p-r))^\times \\ &= w^\times\left(w^\times C(p-r)\right)^\times-\left(w^{\times^2}C(p-r)\right)^\times-w^\times(Cv)^\times+3\left(w^\times Cv\right)^\times-2f^\times+3(Cg)^\times \\ &= \left(w^\times C(p-r)\right)^\times w^\times+2w^\times(Cv)^\times-3(Cv)^\times w^\times-2f^\times+3(Cg)^\times \end{aligned} \tag{72}$$

$$\begin{aligned} B_4-&w^\times\left(2A_3-w^\times\left(6(Cv)^\times-2w^\times(C(p-r))^\times-3(w^\times C(p-r))^\times\right)\right) \\ &= -\left(w^{\times^3}C(p-r)\right)^\times+4\left(w^{\times^2}Cv\right)^\times-3\left(w^\times f\right)^\times+6\left(w^\times Cg\right)^\times \\ &\quad +w^{\times^2}(Cv)^\times+w^\times\left(w^\times Cv\right)^\times-w^\times f^\times+3w^\times(Cg)^\times \\ &\quad +2w^{\times^3}(C(p-r))^\times+2w^{\times^2}\left(w^\times C(p-r)\right)^\times+2w^\times\left(w^{\times^2}C(p-r)\right)^\times-6w^{\times^2}(Cv)^\times-6w^\times\left(w^\times Cv\right)^\times+4w^\times f^\times-6w^\times(Cg)^\times \\ &\quad +6w^{\times^2}(Cv)^\times-2w^{\times^3}(C(p-r))^\times-3w^{\times^2}(w^\times C(p-r))^\times \\ &= w^\times\left(w^{\times^2}C(p-r)\right)^\times-\left(w^{\times^3}C(p-r)\right)^\times+w^{\times^2}(Cv)^\times-5w^\times\left(w^\times Cv\right)^\times+4\left(w^{\times^2}Cv\right)^\times+3w^\times f^\times-3\left(w^\times f\right)^\times-3w^\times(Cg)^\times+6\left(w^\times Cg\right)^\times \\ &= \left(w^{\times^2}C(p-r)\right)^\times w^\times+w^\times(Cv)^\times w^\times-4\left(w^\times Cv\right)^\times w^\times+3f^\times w^\times+3w^\times(Cg)^\times-6(Cg)^\times w^\times \end{aligned} \tag{73}$$

$$
\begin{aligned}
&C_5 - w^\times \left(B_4 - w^\times \left(A_3 - w^\times \left(2(Cv)^\times - w^\times (C(p-r))^\times - (w^\times C(p-r))^\times\right)\right)\right) \\
&= -\left(w^{\times 4} C(p-r)\right)^\times + 5\left(w^{\times 3} Cv\right)^\times - 4\left(w^{\times 2} f\right)^\times + 10\left(w^{\times 2} Cg\right)^\times \\
&\quad + w^\times \left(w^{\times 3} C(p-r)\right)^\times - 3w^\times \left(w^{\times 2} Cv\right)^\times + 2w^\times \left(w^\times f\right)^\times - 2w^\times \left(w^\times Cg\right)^\times \\
&\quad + w^{\times 2} (Cg)^\times \\
&\quad + w^\times \left(w^{\times 3} C(p-r)\right)^\times - 4w^\times \left(w^{\times 2} Cv\right)^\times + 3w^\times \left(w^\times f\right)^\times - 6w^\times \left(w^\times Cg\right)^\times \\
&\quad - w^{\times 3} (Cv)^\times - w^{\times 2} \left(w^\times Cv\right)^\times + w^{\times 2} f^\times - 3w^{\times 2} (Cg)^\times \\
&\quad - w^{\times 4} (C(p-r))^\times - w^{\times 3} \left(w^\times C(p-r)\right)^\times - w^{\times 2} \left(w^{\times 2} C(p-r)\right)^\times + 3w^{\times 3} (Cv)^\times + 3w^{\times 2} \left(w^\times Cv\right)^\times - 2w^{\times 2} f^\times + 3w^{\times 2} (Cg)^\times \\
&\quad - 2w^{\times 3} (Cv)^\times + w^{\times 4} (C(p-r))^\times + w^{\times 3} (w^\times C(p-r))^\times \\
&= w^\times \left(w^{\times 3} C(p-r)\right)^\times - \left(w^{\times 4} C(p-r)\right)^\times - 5w^\times \left(w^{\times 2} Cv\right)^\times + 5\left(w^{\times 3} Cv\right)^\times - w^{\times 2} f^\times + 5w^\times \left(w^\times f\right)^\times - 4\left(w^{\times 2} f\right)^\times \\
&\quad + w^{\times 2} (Cg)^\times - 8w^\times \left(w^\times Cg\right)^\times + 10\left(w^{\times 2} Cg\right)^\times \\
&= \left(w^{\times 3} C(p-r)\right)^\times w^\times - 5\left(w^{\times 2} Cv\right)^\times w^\times - w^\times f^\times w^\times + 4\left(w^\times f\right)^\times w^\times + w^\times (Cg)^\times w^\times - 7\left(w^\times Cg\right)^\times w^\times + 3\left(w^{\times 2} Cg\right)^\times
\end{aligned}
\tag{74}
$$

$$
D_6 = \left(w^{\times 4} C(p-r)\right)^\times w^\times - 6\left(w^{\times 3} Cv\right)^\times w^\times + 5\left(w^{\times 2} f\right)^\times w^\times + 3w^\times \left(w^{\times 2} Cg\right)^\times - 15\left(w^{\times 2} Cg\right)^\times w^\times
$$

$$
D_7 = \left(w^{\times 5} C(p-r)\right)^\times w^\times - 7\left(w^{\times 4} Cv\right)^\times w^\times + 6\left(w^{\times 3} f\right)^\times w^\times + 3w^\times \left(w^{\times 3} Cg\right)^\times - 21\left(w^{\times 3} Cg\right)^\times w^\times
$$

$$
E_8 = -2\left(w^{\times 5} Cv\right)^\times w^\times + 2\left(w^{\times 4} f\right)^\times w^\times - 13\left(w^{\times 4} Cg\right)^\times w^\times
$$

$$
E_9 = -2\left(w^{\times 6} Cv\right)^\times w^\times + 2\left(w^{\times 5} f\right)^\times w^\times - 15\left(w^{\times 5} Cg\right)^\times w^\times
$$

$$
F_{10} = -4\left(w^{\times 6} Cg\right)^\times w^\times
$$

$$
F_{11} = -4\left(w^{\times 7} Cg\right)^\times w^\times
$$

In order to avoid the case where $w = 0$ we evaluate it now and assume afterwards that $w \neq 0$. This gives CASE 0:

$$
\mathcal{O}(x) = \begin{bmatrix} -C & 0 & -(C(p-r))^\times & 0 & 0 & C & 0 \\ 0 & -C & (Cv)^\times & 0 & -(C(p-r))^\times & 0 & 0 \\ 0 & 0 & (Cg)^\times & I & 2(Cv)^\times & 0 & 0 \\ 0 & 0 & 0 & 0 & -2f^\times + 3(Cg)^\times & 0 & 0 \end{bmatrix}
\tag{75}
$$

$$
r(\mathcal{O}(x)) = 9 + r\left(\left[(-2f + 3Cg)^\times\right]\right)
\tag{76}
$$

- $3Cg - 2f = 0 \quad \rightarrow \quad r(\mathcal{O}(x)) = 9$ This correspond to $a = -1/2g$ (acceleration)
- $3Cg - 2f \neq 0 \quad \rightarrow \quad r(\mathcal{O}(x)) = 11$

So from now on we assume $w \neq 0$. For continuing we will have to look at the lower part of the observability matrix. Here the rank depends on the actual motion of the robot. Looking at the column corresponding to the accelerometer bias we extract the following terms and, in a first attempt, try to simplify them (by using row operations). Starting at the forth row this gives:

$$
\begin{aligned}
Z_1 &= \left(w^\times C(p-r)\right)^\times w^\times + 2w^\times (Cv)^\times - 3(Cv)^\times w^\times - 2f^\times + 3(Cg)^\times \\
Z_2 &= \left(w^{\times 2} C(p-r)\right)^\times w^\times + w^\times (Cv)^\times w^\times - 4\left(w^\times Cv\right)^\times w^\times + 3f^\times w^\times + 3w^\times (Cg)^\times - 6(Cg)^\times w^\times \\
Z_3 &= \left(w^{\times 3} C(p-r)\right)^\times w^\times - 5\left(w^{\times 2} Cv\right)^\times w^\times - w^\times f^\times w^\times + 4\left(w^\times f\right)^\times w^\times + w^\times (Cg)^\times w^\times - 7\left(w^\times Cg\right)^\times w^\times + 3\left(w^{\times 2} Cg\right)^\times \\
Z_4 &= \left(w^{\times 4} C(p-r)\right)^\times w^\times - 6\left(w^{\times 3} Cv\right)^\times w^\times + 5\left(w^{\times 2} f\right)^\times w^\times + 3w^\times \left(w^{\times 2} Cg\right)^\times - 15\left(w^{\times 2} Cg\right)^\times w^\times \\
Z_5 &= \left(w^{\times 5} C(p-r)\right)^\times w^\times - 7\left(w^{\times 4} Cv\right)^\times w^\times + 6\left(w^{\times 3} f\right)^\times w^\times + 3w^\times \left(w^{\times 3} Cg\right)^\times - 21\left(w^{\times 3} Cg\right)^\times w^\times \\
Z_6 &= -2\left(w^{\times 5} Cv\right)^\times w^\times + 2\left(w^{\times 4} f\right)^\times w^\times - 13\left(w^{\times 4} Cg\right)^\times w^\times \\
Z_7 &= -2\left(w^{\times 6} Cv\right)^\times w^\times + 2\left(w^{\times 5} f\right)^\times w^\times - 15\left(w^{\times 5} Cg\right)^\times w^\times \\
Z_8 &= -4\left(w^{\times 6} Cg\right)^\times w^\times \\
Z_9 &= -4\left(w^{\times 7} Cg\right)^\times w^\times
\end{aligned}
$$

Scale 8-9, remove 8-9 from 3-7, scale 6-7, remove 6-7 from 3-5:

$$
\begin{aligned}
z_1 &\sim \left(w^{\times} C(p-r)\right)^{\times} w^{\times} + 2w^{\times} (Cv)^{\times} - 3\,(Cv)^{\times} w^{\times} - 2f^{\times} + 3\,(Cg)^{\times} \\
z_2 &\sim \left({w^{\times}}^{2} C(p-r)\right)^{\times} w^{\times} + w^{\times} (Cv)^{\times} w^{\times} - 4\left(w^{\times} Cv\right)^{\times} w^{\times} + 3f^{\times} w^{\times} + 3w^{\times} (Cg)^{\times} - 6\,(Cg)^{\times} w^{\times} \\
z_3 &\sim \left({w^{\times}}^{3} C(p-r)\right)^{\times} w^{\times} - w^{\times} f^{\times} w^{\times} - \left(w^{\times} f\right)^{\times} w^{\times} + w^{\times} (Cg)^{\times} w^{\times} + 3\left({w^{\times}}^{2} Cg\right)^{\times} \\
z_4 &\sim \left({w^{\times}}^{4} C(p-r)\right)^{\times} w^{\times} - \left({w^{\times}}^{2} f\right)^{\times} w^{\times} + 3w^{\times} \left({w^{\times}}^{2} Cg\right)^{\times} \\
z_5 &\sim \left({w^{\times}}^{5} C(p-r)\right)^{\times} w^{\times} - \left({w^{\times}}^{3} f\right)^{\times} w^{\times} + 3w^{\times} \left({w^{\times}}^{3} Cg\right)^{\times} \\
z_6 &\sim \left({w^{\times}}^{3} Cv\right)^{\times} w^{\times} - \left({w^{\times}}^{2} f\right)^{\times} w^{\times} \\
z_7 &\sim \left({w^{\times}}^{2} Cv\right)^{\times} w^{\times} - \left(w^{\times} f\right)^{\times} w^{\times} \\
z_8 &\sim \left({w^{\times}}^{2} Cg\right)^{\times} w^{\times} \\
z_9 &\sim \left(w^{\times} Cg\right)^{\times} w^{\times}
\end{aligned}
$$

Remove 8-9 from 4-5, transform 2:

$$
\begin{aligned}
z_1 &\sim \left(w^{\times} C(p-r)\right)^{\times} w^{\times} + 2w^{\times} (Cv)^{\times} - 3\,(Cv)^{\times} w^{\times} - 2f^{\times} + 3\,(Cg)^{\times} \\
z_2 &\sim \left({w^{\times}}^{2} C(p-r)\right)^{\times} w^{\times} + w^{\times} (Cv)^{\times} w^{\times} - 4\left(w^{\times} Cv\right)^{\times} w^{\times} + 3f^{\times} w^{\times} + 3w^{\times} (Cg)^{\times} - 6\,(Cg)^{\times} w^{\times} \\
z_3 &\sim \left({w^{\times}}^{3} C(p-r)\right)^{\times} w^{\times} - w^{\times} f^{\times} w^{\times} - \left(w^{\times} f\right)^{\times} w^{\times} + w^{\times} (Cg)^{\times} w^{\times} + 3\left({w^{\times}}^{2} Cg\right)^{\times} \\
z_4 &\sim \left({w^{\times}}^{4} C(p-r)\right)^{\times} w^{\times} - \left({w^{\times}}^{2} f\right)^{\times} w^{\times} + 3\left({w^{\times}}^{3} Cg\right)^{\times} \\
z_5 &\sim \left({w^{\times}}^{3} C(p-r)\right)^{\times} w^{\times} - \left(w^{\times} f\right)^{\times} w^{\times} + 3\left({w^{\times}}^{2} Cg\right)^{\times} \\
z_6 &\sim \left({w^{\times}}^{3} Cv\right)^{\times} w^{\times} - \left({w^{\times}}^{2} f\right)^{\times} w^{\times} \\
z_7 &\sim \left({w^{\times}}^{2} Cv\right)^{\times} w^{\times} - \left(w^{\times} f\right)^{\times} w^{\times} \\
z_8 &\sim \left({w^{\times}}^{2} Cg\right)^{\times} w^{\times} \\
z_9 &\sim \left(w^{\times} Cg\right)^{\times} w^{\times}
\end{aligned}
$$

Remove 5 from 3:

$$
\begin{aligned}
z_1 &\sim \left(w^{\times} C(p-r)\right)^{\times} w^{\times} + 2w^{\times} (Cv)^{\times} - 3\,(Cv)^{\times} w^{\times} - 2f^{\times} + 3\,(Cg)^{\times} \\
z_2 &\sim \left({w^{\times}}^{2} C(p-r)\right)^{\times} w^{\times} + w^{\times} (Cv)^{\times} w^{\times} - 4\left(w^{\times} Cv\right)^{\times} w^{\times} + 3f^{\times} w^{\times} + 3\left(w^{\times} Cg\right)^{\times} - 3\,(Cg)^{\times} w^{\times} \\
z_3 &\sim -w^{\times} f^{\times} w^{\times} + w^{\times} (Cg)^{\times} w^{\times} \\
z_4 &\sim \left({w^{\times}}^{4} C(p-r)\right)^{\times} w^{\times} - \left({w^{\times}}^{2} f\right)^{\times} w^{\times} + 3\left({w^{\times}}^{3} Cg\right)^{\times} \\
z_5 &\sim \left({w^{\times}}^{3} C(p-r)\right)^{\times} w^{\times} - \left(w^{\times} f\right)^{\times} w^{\times} + 3\left({w^{\times}}^{2} Cg\right)^{\times} \\
z_6 &\sim \left({w^{\times}}^{3} Cv\right)^{\times} w^{\times} - \left({w^{\times}}^{2} f\right)^{\times} w^{\times} \\
z_7 &\sim \left({w^{\times}}^{2} Cv\right)^{\times} w^{\times} - \left(w^{\times} f\right)^{\times} w^{\times} \\
z_8 &\sim \left({w^{\times}}^{2} Cg\right)^{\times} w^{\times} \\
z_9 &\sim \left(w^{\times} Cg\right)^{\times} w^{\times}
\end{aligned}
$$

Remove omega times 4-5 from 8-9, rescale 8-9, remove 8-9 from 4-5, remove 8 from 2, multiply 2 by omega square, remove omega times 2 from 3, rescale 3:

$$
\begin{aligned}
z_1 &\sim \left(w^{\times} C(p-r)\right)^{\times} w^{\times} + 2w^{\times} (Cv)^{\times} - 3\,(Cv)^{\times} w^{\times} - 2f^{\times} + 3\,(Cg)^{\times} \\
z_2 &\sim \left({w^{\times}}^{4} C(p-r)\right)^{\times} w^{\times} + {w^{\times}}^{3} (Cv)^{\times} w^{\times} - 4\left({w^{\times}}^{3} Cv\right)^{\times} w^{\times} + 3f^{\times} {w^{\times}}^{3} - 3\,(Cg)^{\times} {w^{\times}}^{3} \\
z_3 &\sim {w^{\times}}^{2} (Cv)^{\times} w^{\times} \\
z_4 &\sim \left({w^{\times}}^{4} C(p-r)\right)^{\times} w^{\times} - \left({w^{\times}}^{2} f\right)^{\times} w^{\times} \\
z_5 &\sim \left({w^{\times}}^{3} C(p-r)\right)^{\times} w^{\times} - \left(w^{\times} f\right)^{\times} w^{\times} \\
z_6 &\sim \left({w^{\times}}^{3} Cv\right)^{\times} w^{\times} - \left({w^{\times}}^{2} f\right)^{\times} w^{\times} \\
z_7 &\sim \left({w^{\times}}^{2} Cv\right)^{\times} w^{\times} - \left(w^{\times} f\right)^{\times} w^{\times} \\
z_8 &\sim \left(w^{\times} Cg\right)^{\times} \\
z_9 &\sim \left({w^{\times}}^{2} Cg\right)^{\times}
\end{aligned}
$$

Remove omega times 3 from 2, remove 4 from 2, remove 6 from 2

$$
\begin{aligned}
&z_1 \sim \left(w^{\times} C(p-r)\right)^{\times} w^{\times} + 2w^{\times} (Cv)^{\times} - 3\,(Cv)^{\times} w^{\times} - 2f^{\times} + 3\,(Cg)^{\times} \\
&z_2 \sim -3\left(w^{\times 2} f\right)^{\times} w^{\times} + 3f^{\times} w^{\times 3} - 3\,(Cg)^{\times} w^{\times 3} \\
&z_3 \sim w^{\times} (Cv)^{\times} w^{\times 2} \\
&z_4 \sim \left(w^{\times 4} C(p-r)\right)^{\times} w^{\times} - \left(w^{\times 2} f\right)^{\times} w^{\times} \\
&z_5 \sim \left(w^{\times 3} C(p-r)\right)^{\times} w^{\times} - \left(w^{\times} f\right)^{\times} w^{\times} \\
&z_6 \sim \left(w^{\times} Cv\right)^{\times} w^{\times 3} - \left(w^{\times 2} f\right)^{\times} w^{\times} \\
&z_7 \sim \left(w^{\times} Cv\right)^{\times} w^{\times 2} - \left(w^{\times} f\right)^{\times} w^{\times} \\
&z_8 \sim \left(w^{\times} Cg\right)^{\times} \\
&z_9 \sim \left(w^{\times 2} Cg\right)^{\times}
\end{aligned}
$$

Remove 3 from 6-7, scale 2, scale 6-7

$$
\begin{aligned}
&z_1 \sim \left(w^{\times} C(p-r)\right)^{\times} w^{\times} + 2w^{\times} (Cv)^{\times} - 3\,(Cv)^{\times} w^{\times} - 2f^{\times} + 3\,(Cg)^{\times} \\
&z_2 \sim \left(w^{\times 2} f\right)^{\times} w^{\times} - f^{\times} w^{\times 3} + (Cg)^{\times} w^{\times 3} \\
&z_3 \sim w^{\times} (Cv)^{\times} w^{\times 2} \\
&z_4 \sim \left(w^{\times 4} C(p-r)\right)^{\times} w^{\times} - \left(w^{\times 2} f\right)^{\times} w^{\times} \\
&z_5 \sim \left(w^{\times 3} C(p-r)\right)^{\times} w^{\times} - \left(w^{\times} f\right)^{\times} w^{\times} \\
&z_6 \sim (Cv)^{\times} w^{\times 4} + \left(w^{\times 2} f\right)^{\times} w^{\times} \\
&z_7 \sim (Cv)^{\times} w^{\times 3} + \left(w^{\times} f\right)^{\times} w^{\times} \\
&z_8 \sim \left(w^{\times} Cg\right)^{\times} \\
&z_9 \sim \left(w^{\times 2} Cg\right)^{\times}
\end{aligned}
$$

Remove omega times 6 from 3

$$
\begin{aligned}
&z_1 \sim \left(w^{\times} C(p-r)\right)^{\times} w^{\times} + 2w^{\times} (Cv)^{\times} - 3\,(Cv)^{\times} w^{\times} - 2f^{\times} + 3\,(Cg)^{\times} \\
&z_2 \sim \left(w^{\times 2} f\right)^{\times} w^{\times} - f^{\times} w^{\times 3} + (Cg)^{\times} w^{\times 3} \\
&z_3 \sim 0 \\
&z_4 \sim \left(w^{\times 4} C(p-r)\right)^{\times} w^{\times} - \left(w^{\times 2} f\right)^{\times} w^{\times} \\
&z_5 \sim \left(w^{\times 3} C(p-r)\right)^{\times} w^{\times} - \left(w^{\times} f\right)^{\times} w^{\times} \\
&z_6 \sim (Cv)^{\times} w^{\times 4} + \left(w^{\times 2} f\right)^{\times} w^{\times} \\
&z_7 \sim (Cv)^{\times} w^{\times 3} + \left(w^{\times} f\right)^{\times} w^{\times} \\
&z_8 \sim \left(w^{\times} Cg\right)^{\times} \\
&z_9 \sim \left(w^{\times 2} Cg\right)^{\times}
\end{aligned}
$$

Remove 6 from 2, transform 1

$$
\begin{aligned}
&z_1 \sim -\left(w^{\times 2} C(p-r)\right)^{\times} + w^{\times} \left(w^{\times} C(p-r)\right)^{\times} - w^{\times} (Cv)^{\times} + 3\left(w^{\times} Cv\right)^{\times} - 2f^{\times} + 3\,(Cg)^{\times} \\
&z_2 \sim -(Cv)^{\times} w^{\times 4} - f^{\times} w^{\times 3} + (Cg)^{\times} w^{\times 3} \\
&z_3 \sim 0 \\
&z_4 \sim \left(w^{\times 4} C(p-r)\right)^{\times} w^{\times} - \left(w^{\times 2} f\right)^{\times} w^{\times} \\
&z_5 \sim \left(w^{\times 3} C(p-r)\right)^{\times} w^{\times} - \left(w^{\times} f\right)^{\times} w^{\times} \\
&z_6 \sim (Cv)^{\times} w^{\times 4} + \left(w^{\times 2} f\right)^{\times} w^{\times} \\
&z_7 \sim (Cv)^{\times} w^{\times 3} + \left(w^{\times} f\right)^{\times} w^{\times} \\
&z_8 \sim \left(w^{\times} Cg\right)^{\times} \\
&z_9 \sim \left(w^{\times 2} Cg\right)^{\times}
\end{aligned}
$$

Remove omega times 7 from 2, remove omega times 7 from 6

$$
\begin{aligned}
&z_1 \sim -\left(w^{\times 2} C(p-r)\right)^{\times} + w^{\times}\left(w^{\times} C(p-r)\right)^{\times} - w^{\times}(Cv)^{\times} + 3\left(w^{\times} Cv\right)^{\times} - 2f^{\times} + 3(Cg)^{\times} \\
&z_2 \sim \left(w^{\times} Cv\right)^{\times} w^{\times 3} - f^{\times} w^{\times 3} + (Cg)^{\times} w^{\times 3} \\
&z_3 \sim 0 \\
&z_4 \sim \left(w^{\times 4} C(p-r)\right)^{\times} w^{\times} - \left(w^{\times 2} f\right)^{\times} w^{\times} \\
&z_5 \sim \left(w^{\times 3} C(p-r)\right)^{\times} w^{\times} - \left(w^{\times} f\right)^{\times} w^{\times} \\
&z_6 \sim -\left(w^{\times} Cv\right)^{\times} w^{\times 3} + \left(w^{\times 2} f\right)^{\times} w^{\times} \\
&z_7 \sim (Cv)^{\times} w^{\times 3} + \left(w^{\times} f\right)^{\times} w^{\times} \\
&z_8 \sim \left(w^{\times} Cg\right)^{\times} \\
&z_9 \sim \left(w^{\times 2} Cg\right)^{\times}
\end{aligned}
$$

Now we use the following trick to compute the rank of the matrix of partioned matrices:

$$\forall C : span(C) = Null(A^T) : \tag{77}$$

$$r([AB]) = r(A) + r(C^T B) \tag{78}$$

which can also be applied in the other direction:

$$\forall C : span(C) = Null(A) : \tag{79}$$

$$r\left(\begin{bmatrix} A \\ B \end{bmatrix}\right) = r\left(A^T \;\; B^T\right) \tag{80}$$

$$= r(A^T) + r(C^T B^T) \tag{81}$$

$$= r(A) + r(BC) \tag{82}$$

Furthermore we use the following Lemmas.
Lemma 1:

$$r\left(\left[a^{\times}\right]\right) = \begin{cases} 2, & a \neq 0 \\ 0, & a = 0 \end{cases} \tag{83}$$

Lemma 2:

$$
\begin{aligned}
&a^{\times} b^{\times} c = 0 \\
&\Leftrightarrow (a = 0 \;\vee\; b = 0 \;\vee\; c = 0) \;\vee\; (b \parallel c) \;\vee\; (a \perp b \;\wedge a \perp c)
\end{aligned} \tag{84}
$$

Lemma 3:

$$r\left(\left[a^{\times} b^{\times}\right]\right) = \begin{cases} 2, & a \not\perp b \\ 1, & a \perp b \\ 0, & a = 0 \;\vee\; b = 0 \end{cases} \tag{85}$$

Lemma 4:

$$r\left(\left[a^{\times} b^{\times} a^{\times}\right]\right) = \begin{cases} 2, & a \not\perp b \\ 0, & a \perp b \\ 0, & a = 0 \;\vee\; b = 0 \end{cases} \tag{86}$$

Lemma 5:

$$
\begin{aligned}
&\left(a^{\times} b\right) b = 0 \\
&\Leftrightarrow (a = 0 \;\vee\; b = 0) \;\vee\; (a \parallel b)
\end{aligned} \tag{87}
$$

Lemma 6:

$$r\left(\begin{bmatrix} a^{\times} b^{\times} \\ a^{\times} b^{\times 2} \end{bmatrix}\right) = \begin{cases} 2, & a \not\perp b \\ 0, & a = 0 \;\vee\; b = 0 \end{cases} \tag{88}$$

Now we have to consider different cases. We start with $w \nparallel Cg$ (CASE 1). Here we choose:

$$A = \begin{bmatrix} 0 & \left(w^{\times 2} Cg\right)^{\times} \end{bmatrix} \tag{89}$$

$$B = \begin{bmatrix} 0 & \left(w^{\times} Cg\right)^{\times} \\ 0 & \left(w^{\times} Cv\right)^{\times} w^{\times 3} - f^{\times} w^{\times 3} + (Cg)^{\times} w^{\times 3} \\ 0 & \left(w^{\times 4} C(p-r)\right)^{\times} w^{\times} - \left(w^{\times 2} f\right)^{\times} w^{\times} \\ 0 & \left(w^{\times 3} C(p-r)\right)^{\times} w^{\times} - \left(w^{\times} f\right)^{\times} w^{\times} \\ 0 & -\left(w^{\times} Cv\right)^{\times} w^{\times 3} + \left(w^{\times 2} f\right)^{\times} w^{\times} \\ 0 & (Cv)^{\times} w^{\times 3} + \left(w^{\times} f\right)^{\times} w^{\times} \\ w^{\times}(Cg)^{\times} & -\left(w^{\times 2} C(p-r)\right)^{\times} + w^{\times}\left(w^{\times} C(p-r)\right)^{\times} - w^{\times}(Cv)^{\times} + 3\left(w^{\times} Cv\right)^{\times} - 2f^{\times} + 3(Cg)^{\times} \end{bmatrix} \tag{90}$$

Which gives us:

$$r\left(\begin{bmatrix} A \\ B \end{bmatrix}\right) = r(A) + r(BC) \tag{91}$$

$$= r\left(\begin{bmatrix} 0 & \left(w^{\times 2} Cg\right)^{\times} \end{bmatrix}\right) + r\left(\begin{bmatrix} 0 & \left(w^{\times} Cg\right)^{\times} \\ 0 & \left(w^{\times} Cv\right)^{\times} w^{\times 3} - f^{\times} w^{\times 3} + (Cg)^{\times} w^{\times 3} \\ 0 & \left(w^{\times 4} C(p-r)\right)^{\times} w^{\times} - \left(w^{\times 2} f\right)^{\times} w^{\times} \\ 0 & \left(w^{\times 3} C(p-r)\right)^{\times} w^{\times} - \left(w^{\times} f\right)^{\times} w^{\times} \\ 0 & -\left(w^{\times} Cv\right)^{\times} w^{\times 3} + \left(w^{\times 2} f\right)^{\times} w^{\times} \\ 0 & (Cv)^{\times} w^{\times 3} + \left(w^{\times} f\right)^{\times} w^{\times} \\ w^{\times}(Cg)^{\times} & -\left(w^{\times 2} C(p-r)\right)^{\times} + w^{\times}\left(w^{\times} C(p-r)\right)^{\times} - w^{\times}(Cv)^{\times} + 3\left(w^{\times} Cv\right)^{\times} - 2f^{\times} + 3(Cg)^{\times} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & w^{\times 2} Cg \end{bmatrix}\right) \tag{92}$$

$$= r\left(\begin{bmatrix} 0 & \left(w^{\times 2} Cg\right)^{\times} \end{bmatrix}\right) + r\left(\begin{bmatrix} 0 & \left(w^{\times} Cg\right)^{\times} w^{\times 2} Cg \\ 0 & \# \\ 0 & \# \\ 0 & \# \\ 0 & \# \\ 0 & \# \\ w^{\times}(Cg)^{\times} & \# \end{bmatrix}\right) \tag{93}$$

$$= r\left(\begin{bmatrix} 0 & \left(w^{\times 2} Cg\right)^{\times} \end{bmatrix}\right) + r\left(\begin{bmatrix} \left(w^{\times} Cg\right)^{\times} w^{\times}(w^{\times} Cg) \end{bmatrix}\right) + r\left(\begin{bmatrix} w^{\times}(Cg)^{\times} \end{bmatrix}\right) \tag{94}$$

$$= 2 + 1 + r\left(\begin{bmatrix} w^{\times}(Cg)^{\times} \end{bmatrix}\right) \tag{95}$$

Now we can apply Lemma 2 and 3:

- $w \not\perp Cg \quad \rightarrow \quad r\left(\mathcal{O}(x)\right) = 9 + r\left(\begin{bmatrix} A \\ B \end{bmatrix}\right) = 9 + 2 + 1 + 2 = 14$
- $w \perp Cg \quad \rightarrow \quad r\left(\mathcal{O}(x)\right) = 9 + r\left(\begin{bmatrix} A \\ B \end{bmatrix}\right) = 9 + 2 + 1 + 1 = 13$

For CASE 2 ($w \parallel Cg$), we use the trick in the column form and apply Lemma 3:

$$r([AB]) = r(A) + r(C^T B) \tag{96}$$

$$= r\left(\begin{bmatrix} w^{\times}(Cg)^{\times} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) + r\left(\begin{bmatrix} -w^T\left(w^{\times 2} C(p-r) + 3w^{\times} Cv - 2f\right)^{\times} \\ \left(w^{\times} Cv - f + Cg\right)^{\times} w^{\times 3} \\ \left(w^{\times 4} C(p-r)\right)^{\times} w^{\times} - \left(w^{\times 2} f\right)^{\times} w^{\times} \\ \left(w^{\times 3} C(p-r)\right)^{\times} w^{\times} - \left(w^{\times} f\right)^{\times} w^{\times} \\ -\left(w^{\times 2} Cv - w^{\times} f\right)^{\times} w^{\times 2} \\ (Cv)^{\times} w^{\times 3} + \left(w^{\times} f\right)^{\times} w^{\times} \\ 0 \\ 0 \end{bmatrix}\right) = 2 + r\left(\begin{bmatrix} -w^T\left(w^{\times 2} C(p-r) + 3w^{\times} Cv - 2f\right)^{\times} \\ \left(w^{\times} Cv - f + Cg\right)^{\times} w^{\times 3} \\ \left(w^{\times 4} C(p-r)\right)^{\times} w^{\times} - \left(w^{\times 2} f\right)^{\times} w^{\times} \\ \left(w^{\times 3} C(p-r)\right)^{\times} w^{\times} - \left(w^{\times} f\right)^{\times} w^{\times} \\ -\left(w^{\times 2} Cv - w^{\times} f\right)^{\times} w^{\times 2} \\ (Cv)^{\times} w^{\times 3} + \left(w^{\times} f\right)^{\times} w^{\times} \end{bmatrix}\right) \tag{97}$$

Here we look at three subcases. For CASE 2.A ($w^{\times} Cv - f = -Cg$) we have (it directly follows that $w^{\times 2} Cv = w^{\times} f$):

$$r([AB]) = r(A) + r(C^T B) \tag{98}$$

$$= 2 + r\left(\begin{bmatrix} -w^T \left(w^{\times 2} C(p-r) + 3w^{\times} Cv - 2f\right)^{\times} \\ 0 \\ \left(w^{\times 4} C(p-r)\right)^{\times} w^{\times} - \left(w^{\times 2} f\right)^{\times} w^{\times} \\ \left(w^{\times 3} C(p-r)\right)^{\times} w^{\times} - \left(w^{\times} f\right)^{\times} w^{\times} \\ \left(w^{\times} Cg\right)^{\times} w^{\times 2} \\ (Cv)^{\times} w^{\times 3} + \left(w^{\times} f\right)^{\times} w^{\times} \end{bmatrix}\right) = 2 + r\left(\begin{bmatrix} -w^T \left(w^{\times 2} C(p-r) + w^{\times} Cv\right)^{\times} \\ 0 \\ \left(w^{\times 4} C(p-r)\right)^{\times} w^{\times} - \left(w^{\times 3} Cv\right)^{\times} w^{\times} \\ \left(w^{\times 3} C(p-r)\right)^{\times} w^{\times} - \left(w^{\times 2} Cv\right)^{\times} w^{\times} \\ 0 \\ (Cv)^{\times} w^{\times 3} + \left(w^{\times 2} Cv\right)^{\times} w^{\times} \end{bmatrix}\right) \tag{99}$$

$$= 2 + r\left(\begin{bmatrix} -w^T \left(w^{\times 2} C(p-r) + w^{\times} Cv\right)^{\times} \\ \left(w^{\times 2} C(p-r) - w^{\times} Cv\right)^{\times} w^{\times 3} \\ \left(w^{\times 2} C(p-r) - w^{\times} Cv\right)^{\times} w^{\times 2} \\ w^{\times 2} (Cv)^{\times} w^{\times} \end{bmatrix}\right) \tag{100}$$

For $Cv \perp w$ (CASE 2.A.i) we have (including $Cv = 0$ and resulting in $w^{\times} (Cv)^{\times} w^{\times} = 0$):

$$r([AB]) = 2 + r\left(\begin{bmatrix} -w^T \left(w^{\times 2} C(p-r) + w^{\times} Cv\right)^{\times} \\ \left(w^{\times 2} C(p-r) - w^{\times} Cv\right)^{\times} w^{\times} \\ \left(w^{\times 2} C(p-r) - w^{\times} Cv\right)^{\times} w^{\times 2} \end{bmatrix}\right) \tag{101}$$

$$= 2 + r\left(\begin{bmatrix} -w^T \left(w^{\times 2} C(p-r) + w^{\times} Cv\right)^{\times} \\ \left(w^{\times 2} C(p-r) - w^{\times} Cv\right)^{\times} w^{\times} \\ \left(w^{\times 2} C(p-r) - w^{\times} Cv\right)^{\times} w^{\times 2} \end{bmatrix}\right) \tag{102}$$

For $w^{\times} C(p-r) - Cv \not\parallel w$ we can apply Lemma 6 and get

$$r\left(\mathcal{O}(x)\right) = 9 + r([AB]) = 9 + 2 + 2 = 13 \tag{103}$$

For $w^{\times} C(p-r) - Cv \parallel w$ it directly follows that $w^{\times} C(p-r) = Cv$ (since both are perpendicular to w). We have to look at the first row (the lower two rows become 0):

$$r([AB]) = 2 + r\left(\left[-2w^T \left(w^{\times} (w^{\times} C(p-r))\right)^{\times}\right]\right) \tag{104}$$

and thus we have

- $C(p-r) \parallel w \quad \rightarrow \quad r\left(\mathcal{O}(x)\right) = 9 + r([AB]) = 9 + 2 = 11$ (including $C(p-r) = 0$)
- $C(p-r) \not\parallel w \quad \rightarrow \quad r\left(\mathcal{O}(x)\right) = 9 + r([AB]) = 9 + 2 + 1 = 12$

Looking at the last subcase $Cv \not\perp w$ (CASE 2.A.ii) we can directly apply Lemma 4:

$$r\left(\mathcal{O}(x)\right) = 9 + r([AB]) = 9 + 2 + 2 = 13 \tag{105}$$

For CASE 2.B ($w^{\times} Cv - f + Cg \perp w$) we can apply Lemma 3 (it also directly follows that $f \neq 0$):

$$r([AB]) = r(A) + r(C^T B) \tag{106}$$

$$= 2 + r\left(\begin{bmatrix} -w^T \left(w^{\times 2} C(p-r) + 3w^{\times} Cv - 2f\right)^{\times} \\ \left(w^{\times} Cv - f + Cg\right)^{\times} w^{\times 3} \\ \left(w^{\times 4} C(p-r)\right)^{\times} w^{\times} - \left(w^{\times 2} f\right)^{\times} w^{\times} \\ \left(w^{\times 3} C(p-r)\right)^{\times} w^{\times} - \left(w^{\times} f\right)^{\times} w^{\times} \\ -\left(w^{\times 2} Cv - w^{\times} f\right)^{\times} w^{\times 2} \\ (Cv)^{\times} w^{\times 3} + \left(w^{\times} f\right)^{\times} w^{\times} \end{bmatrix}\right) \tag{107}$$

$$= 2 + r\left(\left[\left(w^{\times} Cv - f + Cg\right)^{\times} w^{\times}\right]\right) + r\left(\begin{bmatrix} -w^T \left(w^{\times 2} C(p-r) + 3w^{\times} Cv - 2f\right)^{\times} \\ \left(w^{\times 4} C(p-r)\right)^{\times} w^{\times} - \left(w^{\times 2} f\right)^{\times} w^{\times} \\ \left(w^{\times 3} C(p-r)\right)^{\times} w^{\times} - \left(w^{\times} f\right)^{\times} w^{\times} \\ -\left(w^{\times 2} Cv - w^{\times} f\right)^{\times} w^{\times 2} \\ (Cv)^{\times} w^{\times 3} + \left(w^{\times} f\right)^{\times} w^{\times} \end{bmatrix} \begin{bmatrix} w & \left(w^{\times} Cv - f + Cg\right)^{\times} w \end{bmatrix}\right) \tag{108}$$

$$= 2 + 1 + r\left(\begin{bmatrix} -w^T \left(w^{\times 2} C(p-r) + 3w^{\times} Cv - 2f\right)^{\times} \left(w^{\times} Cv - f\right)^{\times} w \\ \left(w^{\times 4} C(p-r)\right)^{\times} w^{\times} \left(w^{\times} Cv - f\right)^{\times} w - \left(w^{\times 2} f\right)^{\times} w^{\times} \left(w^{\times} Cv - f\right)^{\times} w \\ \left(w^{\times 3} C(p-r)\right)^{\times} w^{\times} \left(w^{\times} Cv - f\right)^{\times} w - \left(w^{\times} f\right)^{\times} w^{\times} \left(w^{\times} Cv - f\right)^{\times} w \\ 0 \\ (Cv)^{\times} w^{\times 4} \left(w^{\times} Cv - f\right) + \left(w^{\times} f\right)^{\times} w^{\times 2} \left(w^{\times} Cv - f\right) \end{bmatrix}\right) \tag{109}$$

The last term can be evaluated to:

$$(Cv)^\times w^{\times 4}\left(w^\times Cv - f\right) + \left(w^\times f\right)^\times w^{\times 2}\left(w^\times Cv - f\right) = \left(-\|w\|^2 (Cv)^\times + \left(w^\times f\right)^\times\right) w^{\times 2}\left(w^\times Cv - f\right) \tag{110}$$

$$= \left(-\|w\|^2 Cv + w^\times f\right)^\times w^{\times 2}\left(w^\times Cv - f\right) \tag{111}$$

$$= \left(-\|w\|^2 Cv + w^\times f\right)^\times w^\times\left(w^{\times 2} Cv - w^\times f\right) \tag{112}$$

For $Cv = 0$ we get

$$(Cv)^\times w^{\times 4}\left(w^\times Cv - f\right) + \left(w^\times f\right)^\times w^{\times 2}\left(w^\times Cv - f\right) = \left(w^\times f\right)^\times w^\times\left(w^\times f\right) \tag{113}$$

and applying Lemma 2 yields (f cannot parallel to w whitout getting CASE 2.A)::

$$r\left(\mathcal{O}(x)\right) = 9 + r([AB]) = 9 + 2 + 1 + 1 = 13 \tag{114}$$

For $Cv \not\perp w$ we can check the perpendicularity condition of Lemma 2:

$$w^T\left(-\|w\|^2 Cv + w^\times f\right) = -\|w\|^2 w^T Cv \neq 0 \tag{115}$$

and thus again get

$$r\left(\mathcal{O}(x)\right) = 9 + r([AB]) = 9 + 2 + 1 + 1 = 13. \tag{116}$$

For $Cv \perp w$ both perpenducalirity conditions of Lemma 2 yield:

$$w^T\left(-\|w\|^2 Cv + w^\times f\right) = -\|w\|^2 w^T Cv = 0 \tag{117}$$

$$\left(v^T C^T w^{\times 2} + f^T w^\times\right)\left(-\|w\|^2 Cv + w^\times f\right) = -v^T C^T w^{\times 2}\|w\|^2 Cv - f^T w^\times \|w\|^2 Cv + v^T C^T w^{\times 2} w^\times f + f^T w^\times w^\times f \tag{118}$$

$$= v^T C^T w^{\times 2} w^{\times 2} Cv + f^T w^\times w^\times f \tag{119}$$

$$= \|w^{\times 2} Cv\|^2 - \|w^\times f\|^2 \tag{120}$$

$$= \|w\|^4 \|Cv\|^2 - \|w^\times f\|^2 \tag{121}$$

And thus we have a singularity for

$$\frac{\|w^{\times 2} Cv\|}{\|w\|^2} = \frac{\|w^\times Cv\|}{\|w\|} = \|Cv\| = \frac{\|w^\times f\|}{\|w\|^2} \quad \wedge \quad Cv \perp w \tag{122}$$

Looking at the reminding elements we get:

$$r([AB]) = 3 + r\left(\begin{bmatrix} -w^T\left(w^{\times 2} C(p-r) + 3w^\times Cv - 2f\right)^\times \left(w^\times Cv - f\right)^\times w \\ \left(w^{\times 3} C(p-r) - w^\times f\right)^\times w^\times \left(w^\times Cv - f\right) \\ \left(w^{\times 3} C(p-r) - w^\times f\right)^\times w^{\times 2}\left(w^\times Cv - f\right) \end{bmatrix}\right) \tag{123}$$

Now we can exclude the case where $w^\times Cv - f \parallel w$ (otherwise we obtain $w^\times Cv - f = -Cg$ which is CASE 2.A). We again require two cases. In the first case we state $w^{\times 2} C(p-r) - f \parallel w$ (including the case $w^{\times 2} C(p-r) - f = 0$) and thus obtain:

$$r([AB]) = 3 + r\left(\left[-w^T\left(w^{\times 2} C(p-r) + 3w^\times Cv - 2f\right)^\times \left(w^\times Cv - f\right)^\times w\right]\right) \tag{124}$$

$$= 3 + r\left(\left[-w^T\left(3w^\times Cv - f\right)^\times \left(w^\times Cv - f\right)^\times w\right]\right) \tag{125}$$

$$= 3 + r\left(\left[-\left(3w^\times Cv - f\right)^T w^{\times 2}\left(w^\times Cv - f\right)\right]\right) \tag{126}$$

$$= 3 + r\left(\left[3v^T C^T w^{\times 4} Cv + 4f^T w^{\times 3} Cv - f^T w^{\times 2} f\right]\right) \tag{127}$$

$$= 3 + r\left(\left[3\|w^{\times 2} Cv\|^2 - 4\left(w^\times f\right)^T\left(w^{\times 2} Cv\right) + \|w^\times f\|^2\right]\right) \tag{128}$$

$$= 3 + r\left(\left[4\|w^\times f\|^2 - 4\left(w^\times f\right)^T\left(w^{\times 2} Cv\right)\right]\right) \tag{129}$$

$$= 3 + r\left(\left[4(1 - \cos(\theta))\|w^\times f\|^2\right]\right) = 4 \tag{130}$$

Where we used (122) and where θ is the angle between $w^\times f$ and $w^{\times 2} Cv$. Thus we only get 0 on the right hand side if $w^\times f = w^{\times 2} Cv$ which means that $w^\times (f - w^\times Cv) = 0$ and again leads to the case $w^\times Cv - f = -Cg$. For the second case we set $a = w^{\times 2} C(p-r) - f \nparallel w$ and obtain

$$r([AB]) = 3 + r\left(\begin{bmatrix} -w^T\left(w^{\times 2} C(p-r) + 3w^\times Cv - 2f\right)^\times \left(w^\times Cv - f\right)^\times w \\ \left(w^\times a\right)^\times w^\times \left(w^\times Cv - f\right) \\ \left(w^\times a\right)^\times w^{\times 2}\left(w^\times Cv - f\right) \end{bmatrix}\right) \tag{131}$$

where considering Lemma 6 (the nullspace of the two lower rows is spanned by w only) leads to $r([AB]) = 4$

For CASE 2.C ($w^\times Cv - f + Cg \not\perp w$) we can directly apply Lemma 3:

$$r([AB]) = r(A) + r(C^T B) \tag{132}$$

$$= 2 + r\left(\begin{bmatrix} -w^T\left(w^{\times 2}C(p-r) + 3w^\times Cv - 2f\right)^\times \\ \left(w^\times Cv - f + Cg\right)^\times w^{\times 3} \\ \left(w^{\times 4}C(p-r)\right)^\times w^\times - \left(w^{\times 2}f\right)^\times w^\times \\ \left(w^{\times 3}C(p-r)\right)^\times w^\times - \left(w^\times f\right)^\times w^\times \\ -\left(w^{\times 2}Cv - w^\times f\right)^\times w^{\times 2} \\ (Cv)^\times w^{\times 3} + \left(w^\times f\right)^\times w^\times \\ 0 \\ 0 \end{bmatrix}\right) \tag{133}$$

$$= 2 + r\left(\left[\left(w^\times Cv - f + Cg\right)^\times w^\times\right]\right) + r\left(\begin{bmatrix} -w^T\left(w^{\times 2}C(p-r) + 3w^\times Cv - 2f\right)^\times \\ \left(w^{\times 4}C(p-r)\right)^\times w^\times - \left(w^{\times 2}f\right)^\times w^\times \\ \left(w^{\times 3}C(p-r)\right)^\times w^\times - \left(w^\times f\right)^\times w^\times \\ -\left(w^{\times 2}Cv - w^\times f\right)^\times w^{\times 2} \\ (Cv)^\times w^{\times 3} + \left(w^\times f\right)^\times w^\times \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} w \end{bmatrix}\right) \tag{134}$$

$$= 2 + 2 + 0 = 4 \tag{135}$$

$$r\left(\mathcal{O}(x)\right) = 9 + r([AB]) = 9 + 2 + 2 = 13 \tag{136}$$

All the cases are summarized in the table I. It contains all possible motions. The first row corresponds to motions with no rotational velocity. In the case the robot is accelerated with -1/2 g the rank loss increases to 5. The second row states that if the rotational velocity is perpendicular to the gravity axis we have a rank loss of 1. The third row corresponds to rotations around the gravity only. If, additionaly the motion is circular around the footpoint then the rank loss increases to 2 or 3 if the foot is aligned along the gravity axis. The last column states that in all other cases all states are fully observable up to the global position and yaw angle.

$w = 0$	$w \perp Cg$	$w \parallel Cg$	Rank deficiency
x	x	x	5 $(a = -1/2g)$ 3 $(a \neq -1/2g)$
	x		1
		x	3 $(f = w^\times Cv + Cg \ \wedge\ w^\times C(p-r) = Cv \ \wedge\ (p-r) \parallel g)$ 2 $(f = w^\times Cv + Cg \ \wedge\ w^\times C(p-r) = Cv \ \wedge\ (p-r) \nparallel g)$ 1 o.w.
			0

TABLE I
RANK DEFICIENCY IN DEPENDENCE OF INPUT PARAMETERS.

<table>
<tr><td colspan="5"></td><td>Rank deficiency</td></tr>
<tr><td rowspan="5">$w \neq 0$</td><td rowspan="4">$w \not\perp Cg$</td><td colspan="3">$w \nparallel Cg$</td><td>0</td></tr>
<tr><td rowspan="3">$w \parallel Cg$</td><td colspan="2">$f \neq w^\times Cv + Cg \ \vee\ w^\times C(p-r) \neq Cv$</td><td>1</td></tr>
<tr><td rowspan="2">$f = w^\times Cv + Cg \ \wedge\ w^\times C(p-r) = Cv$</td><td>$(p-r) \nparallel g$</td><td>2</td></tr>
<tr><td>$(p-r) \parallel g$</td><td>3</td></tr>
<tr><td colspan="4">$w \perp Cg$</td><td>1</td></tr>
<tr><td colspan="3" rowspan="2">$w = 0$</td><td colspan="2">$a \neq -1/2g$</td><td>3</td></tr>
<tr><td colspan="2">$a = -1/2g$</td><td>5</td></tr>
</table>

TABLE II
RANK DEFICIENCY IN DEPENDENCE OF INPUT PARAMETERS.

<table>
<tr><td colspan="5">$w \neq 0$</td><td colspan="2" rowspan="3">$w = 0$</td></tr>
<tr><td colspan="4">$w \not\perp Cg$</td><td rowspan="4">$w \perp Cg$</td></tr>
<tr><td rowspan="3">$w \nparallel Cg$</td><td colspan="3">$w \parallel Cg$</td></tr>
<tr><td rowspan="2">$f \neq w^\times Cv + Cg$
$\vee\ w^\times C(p-r) \neq Cv$</td><td colspan="2">$f = w^\times Cv + Cg \ \wedge\ w^\times C(p-r) = Cv$</td><td rowspan="2">$a \neq -1/2g$</td><td rowspan="2">$a = -1/2g$</td></tr>
<tr><td>$(p-r) \nparallel g$</td><td>$(p-r) \parallel g$</td></tr>
<tr><td>0</td><td>1</td><td>2</td><td>3</td><td>1</td><td>3</td><td>5</td></tr>
</table>

TABLE III
RANK DEFICIENCY IN DEPENDENCE OF INPUT PARAMETERS.

2) Planar Foot: For the planar foot we have additional measurements from the rotational constraint of the contact:

$$\mathcal{L}_f^0 h(x) = z \otimes q^{-1} \tag{137}$$

$$\nabla \mathcal{L}_f^0 h(x) = \begin{bmatrix} 0 & 0 & -C(z)C^T & 0 & 0 & 0 & I \end{bmatrix} \tag{138}$$

$$\mathcal{L}_f^1 h(x) = C(z)C^T \hat{\omega} \tag{139}$$

$$\nabla \mathcal{L}_f^1 h(x) = \begin{bmatrix} 0 & 0 & C(z)C^T \hat{\omega}^\times & 0 & -C(z)C^T & 0 & -(C(z)C^T \hat{\omega})^\times \end{bmatrix} \tag{140}$$

$$\mathcal{L}_f^2 h(x) = 0 \tag{141}$$

$$\tag{142}$$

With this the gyroscope bias gets fully observable and the above Observability matrix can be augmented and simplified to:

$$\mathcal{O}(x) = \begin{bmatrix} -C & 0 & -(C(p-r))^\times & 0 & 0 & C & 0 \\ 0 & -C & (Cv)^\times & 0 & 0 & 0 & 0 \\ 0 & 0 & (Cg)^\times & I & 0 & 0 & 0 \\ 0 & 0 & -\hat{\omega}^\times (Cg)^\times & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -C(z)C^T & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & -C(z)C^T & 0 & 0 \end{bmatrix} \tag{143}$$

Where the unobservable subspace is spanned by:

$$\mathcal{U}(x) = \begin{bmatrix} I & 0 & 0 & 0 & 0 & I & 0 \\ \left(r^\times g\right)^T & \left(v^\times g\right)^T & (Cg)^T & 0 & 0 & \left(p^\times g\right)^T & (C(z)g)^T \end{bmatrix}^T \tag{144}$$

Which correspond to a global translation or yaw rotation of the states. If for instance we have $w = 0$ the unobservable subspace increases:

$$\mathcal{U}(x) = \begin{bmatrix} I & 0 & 0 & 0 & 0 & I & 0 \\ \left(r^\times\right)^T & \left(v^\times\right)^T & (C)^T & -\left(Cg^\times\right)^T & 0 & \left(p^\times\right)^T & (C(z))^T \end{bmatrix}^T \tag{145}$$

where the bottom line now corresponds to the full 3D rotation of the system. In summary this gives:

$w = 0$	$w \perp Cg$	Rank deficiency
x	x	2
	x	1
		0

TABLE IV
RANK DEFICIENCY IN DEPENDENCE OF INPUT PARAMETERS.

3) Two Point Feet: In the case were we have contact with two point feet additional states and measurements get avaliable. Analytically this is basically duplicating the observability matrix. If the contact points overlay each other (which is not physicaly feasible), we obtain the same observability characteristics as for the single contact case. From here on we assume that $\Delta p = p_1 - p_2 \neq 0$. Duplicating and simplifying the first four rows yields:

$$\begin{bmatrix} -C & 0 & -(C(p_2 - r))^\times & 0 & 0 & 0 & C \\ 0 & 0 & 0 & 0 & (C\Delta p)^\times & 0 & 0 \\ 0 & 0 & 0 & 0 & (C\Delta p)^\times w^\times & 0 & 0 \\ 0 & 0 & 0 & 0 & \left(w^\times C\Delta p\right)^\times w^\times & 0 & 0 \end{bmatrix} \tag{146}$$

While the first row compensates for the increase in states, the 3 lower rows further reduce the unobservable space. Again we start by looking at the case $w = 0$:

$$\mathcal{O}(x) = \begin{bmatrix} -C & 0 & -(C(p_2 - r))^\times & 0 & 0 & 0 & C \\ -C & 0 & -(C(p-r))^\times & 0 & 0 & C & 0 \\ 0 & -C & (Cv)^\times & 0 & -(C(p-r))^\times & 0 & 0 \\ 0 & 0 & (Cg)^\times & I & 2(Cv)^\times & 0 & 0 \\ 0 & 0 & 0 & 0 & -2f^\times + 3\,(Cg)^\times & 0 & 0 \\ 0 & 0 & 0 & 0 & (C\Delta p)^\times & 0 & 0 \end{bmatrix} \tag{147}$$

$$r\left(\mathcal{O}(x)\right) = 12 + r\left(\left[(C\Delta p)^\times\right]\right) + r\left(\left[(-2f + 3Cg)^\times \; C\Delta p\right]\right) \tag{148}$$

- $3Cg - 2f \parallel C\Delta p \quad \rightarrow \quad r\left(\mathcal{O}(x)\right) = 14$
- $3Cg - 2f \not\parallel C\Delta p \quad \rightarrow \quad r\left(\mathcal{O}(x)\right) = 15$

Continuing with the same procedure as above yields (where most duplicates could be eliminated):

$$
\begin{aligned}
Z_1 &\sim \left(w^{\times} C(p_1 - r)\right)^{\times} w^{\times} - w^{\times} (Cv)^{\times} + 3\left(w^{\times} Cv\right)^{\times} - 2f^{\times} + 3(Cg)^{\times} \\
Z_2 &\sim \left(w^{\times} Cv\right)^{\times} {w^{\times}}^3 - f^{\times} {w^{\times}}^3 + (Cg)^{\times} {w^{\times}}^3 \\
Z_3 &\sim 0 \\
Z_4 &\sim \left({w^{\times}}^4 C(p_1 - r)\right)^{\times} w^{\times} - \left({w^{\times}}^2 f\right)^{\times} w^{\times} \\
Z_5 &\sim \left({w^{\times}}^3 C(p_1 - r)\right)^{\times} w^{\times} - \left(w^{\times} f\right)^{\times} w^{\times} \\
Z_6 &\sim -\left(w^{\times} Cv\right)^{\times} {w^{\times}}^3 + \left({w^{\times}}^2 f\right)^{\times} w^{\times} \\
Z_7 &\sim (Cv)^{\times} {w^{\times}}^3 + \left(w^{\times} f\right)^{\times} w^{\times} \\
Z_8 &\sim \left(w^{\times} Cg\right)^{\times} \\
Z_9 &\sim \left({w^{\times}}^2 Cg\right)^{\times} \\
\tilde{Z}_{-1} &\sim (C\Delta p)^{\times} \\
\tilde{Z}_0 &\sim (C\Delta p)^{\times} w^{\times} \\
\tilde{Z}_1 &\sim \left(w^{\times} C\Delta p\right)^{\times} w^{\times}
\end{aligned}
$$

We don't have to look at the case $w \nparallel Cg \ \wedge \ w \not\perp Cg$, since we already got the maximal oservability in the single foot case. The case $w \perp Cg$ also leads to the same results as the single foot case (one rank loss):

- $w \nparallel Cg \ \wedge \ w \not\perp Cg \quad \rightarrow \quad r(\mathcal{O}(x)) = 17$
- $w \perp Cg \quad \rightarrow \quad r(\mathcal{O}(x)) = 16$

For $w \parallel Cg$, we get:

$$
r([AB]) = r(A) + r(C^T B) \tag{149}
$$

$$
= r\left(\begin{bmatrix} w^{\times}(Cg)^{\times} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) + r\left(\begin{bmatrix} -w^T\left({w^{\times}}^2 C(p_1 - r) + 3w^{\times} Cv - 2f\right)^{\times} \\ \left(w^{\times} Cv - f + Cg\right)^{\times} {w^{\times}}^3 \\ \left({w^{\times}}^4 C(p_1 - r)\right)^{\times} w^{\times} - \left({w^{\times}}^2 f\right)^{\times} w^{\times} \\ \left({w^{\times}}^3 C(p_1 - r)\right)^{\times} w^{\times} - \left(w^{\times} f\right)^{\times} w^{\times} \\ -\left({w^{\times}}^2 Cv - w^{\times} f\right)^{\times} {w^{\times}}^2 \\ (Cv)^{\times} {w^{\times}}^3 + \left(w^{\times} f\right)^{\times} w^{\times} \\ (C\Delta p)^{\times} \\ (C\Delta p)^{\times} w^{\times} \\ \left(w^{\times} C\Delta p\right)^{\times} w^{\times} \end{bmatrix}\right) \tag{150}
$$

$$
= 2 + r\left(\left[(C\Delta p)^{\times}\right]\right) + r\left(\begin{bmatrix} -w^T\left({w^{\times}}^2 C(p_1 - r) + 3w^{\times} Cv - 2f\right)^{\times} \\ \left(w^{\times} Cv - f + Cg\right)^{\times} {w^{\times}}^3 \\ \left({w^{\times}}^4 C(p_1 - r)\right)^{\times} w^{\times} - \left({w^{\times}}^2 f\right)^{\times} w^{\times} \\ \left({w^{\times}}^3 C(p_1 - r)\right)^{\times} w^{\times} - \left(w^{\times} f\right)^{\times} w^{\times} \\ -\left({w^{\times}}^2 Cv - w^{\times} f\right)^{\times} {w^{\times}}^2 \\ (Cv)^{\times} {w^{\times}}^3 + \left(w^{\times} f\right)^{\times} w^{\times} \\ (C\Delta p)^{\times} w^{\times} \\ \left(w^{\times} C\Delta p\right)^{\times} w^{\times} \end{bmatrix} \begin{bmatrix} C\Delta p \end{bmatrix}\right) \tag{151}
$$

$$
= 2 + 2 + r\left(\begin{bmatrix} -w^T\left({w^{\times}}^2 C(p_1 - r) + 3w^{\times} Cv - 2f\right)^{\times} C\Delta p \\ \left(w^{\times} Cv - f + Cg\right)^{\times} {w^{\times}}^3 C\Delta p \\ \left({w^{\times}}^4 C(p_1 - r)\right)^{\times} w^{\times} C\Delta p - \left({w^{\times}}^2 f\right)^{\times} w^{\times} C\Delta p \\ \left({w^{\times}}^3 C(p_1 - r)\right)^{\times} w^{\times} C\Delta p - \left(w^{\times} f\right)^{\times} w^{\times} C\Delta p \\ -\left({w^{\times}}^2 Cv - w^{\times} f\right)^{\times} {w^{\times}}^2 C\Delta p \\ (Cv)^{\times} {w^{\times}}^3 C\Delta p + \left(w^{\times} f\right)^{\times} w^{\times} C\Delta p \\ (C\Delta p)^{\times} w^{\times} C\Delta p \\ 0 \end{bmatrix}\right) \tag{152}
$$

Which directly yields (Lemma 2):

- $w \nparallel \Delta p \quad \rightarrow \quad r(\mathcal{O}(x)) = 17$
- $w \parallel \Delta p \quad \rightarrow \quad r(\mathcal{O}(x)) = 16$

Summarized in a table this gives:

$w \neq 0$				$w = 0$	
$w \not\perp Cg$			$w \perp Cg$		
$w \nparallel Cg$	$w \parallel Cg$			$3Cg - 2f \nparallel C\Delta p$	$3Cg - 2f \parallel C\Delta p$
	$w \nparallel \Delta p$	$w \nparallel \Delta p$			
0	0	1	1	2	3

TABLE V
RANK DEFICIENCY IN DEPENDENCE OF INPUT PARAMETERS.

4) Two Flat Feet: For the case of two flat feet, we can quicly prove that it can be reduced to the 1 flat foot case.

$$\mathcal{O}(x) = \begin{bmatrix} -C & 0 & -(C(p_1 - r))^\times & 0 & 0 & C & 0 & 0 & 0 \\ -C & 0 & -(C(p_2 - r))^\times & 0 & 0 & 0 & C & 0 & 0 \\ 0 & -C & (Cv)^\times & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (Cg)^\times & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\hat{\omega}^\times (Cg)^\times & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -C(z_1)C^T & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & -C(z_1)C^T & 0 & 0 & 0 & 0 \\ 0 & 0 & -C(z_2)C^T & 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & -C(z_2)C^T & 0 & 0 & 0 & 0 \end{bmatrix} \tag{153}$$

In summary this gives:

		Rank deficiency
$w \neq 0$	$w \not\perp Cg$	0
	$w \perp Cg$	1
$w = 0$		2

TABLE VI
RANK DEFICIENCY IN DEPENDENCE OF INPUT PARAMETERS.

III. Observability Analysis Summary

This section discusses the observability characteristics of the proposed approach and analyses the gain obtained by the inclusion of the flat foot constraint. The unobservable subspace, which, informally, describes all directions along which disturbances cannot be observed at the system output, can change depending on the motion of the system. In order to evaluate this for the proposed filter, the observability analysis approach employed in [1] is applied. The full derivation is not in the scope of this paper, but the obtained results are summarized in table VII, VIII, and IX. The analysis is extensive and displays all possible singularities and corresponding rank losses (RL). Since the absolute position and yaw angle are inherently unobservable for the presented system, the rank loss represents an increase of the unobservable subspace with respect to that case. If the rank loss is not 0, i.e., if the unobservable subspace increases, additional direction become unobservable, meaning that disturbances along this directions can not be observed at the output of the system.

Rotation	Acceleration/Velocity	Foothold	RL
$w = 0$	$a = -1/2g$	$*$	5
	$a \neq -1/2g$	$*$	3
$w \perp Cg$	$*$	$*$	1
$w \parallel Cg$	$\wedge$ $a = (C^T w) \times v$, $v = (C^T w) \times (r-p)$	$(r-p) \parallel g$	3
		$(r-p) \nparallel g$	2
	$\vee$ $a \neq (C^T w) \times v$, $v \neq (C^T w) \times (r-p)$	$*$	1
$\wedge$ $w \not\perp Cg$, $w \nparallel Cg$	$*$	$*$	0

TABLE VII
RANK DEFICIENCY FOR A SINGLE POINT FOOT CONTACT.

Table VII describes the rank deficiency for the case where a single point foot is in contact with the ground. The rows of the table can be interpreted as follows. The top row ($w = 0$) describes the case where there is no rotational motion. Depending on the actual acceleration the rank loss will be 3 or 5. The second row ($w \perp Cg$) states that whenever there is a rotational motion where the corresponding rotation vector is perpendicular to the gravity then the rank loss is 1 (interestingly this singularity arises within all other scenarios as well). The third row ($w \parallel Cg$) describes the slightly more complicated case when there is a rotation around the gravity axis only. In this case one has to distinguish the subcase where the robot performs a circular motion around the gravity axis through the footpoint. If this is the case the rank loss increases from 1 to 2. If in addition the vector between the IMU and the footpoint is aligned with the gravity axis the rank loss even increases to 3. Finally the last row states that if the rotational motion is neither aligned with the gravity axis nor perpendicular to it then the system does not exhibit any additional rank deficiency.

Rotation	Footholds	RL
$w = 0$	$2a + g \parallel \Delta p$	3
	$2a + g \nparallel \Delta p$	2
$w \perp Cg$	$*$	1
$w \parallel Cg$	$g \parallel \Delta p$	1
	$g \nparallel \Delta p$	0
$\wedge$ $w \not\perp Cg$, $w \nparallel Cg$	$*$	0

TABLE VIII
RANK DEFICIENCY FOR TWO POINT FOOT CONTACTS.

In comparison to the point foot case, the observability analysis for the flat foot case is significantly simpler to compute and interpret. If looking at table IX, one can observe that the rank loss is depending on the current rotational velocity only. If there is no rotational velocity the rank loss is 2, if it is perpendicular to the gravity axis then the rank loss is 1. For all other cases the rank loss is 0, i.e., only absolute position and yaw angle are unobservable. When comparing this to the point foot case, there is a significant reduction of the rank loss, which is due to the additional information which can be extracted from the rotational constraint of the flat foot. For instance the maximal rank loss is reduced from 5 to 2 for no rotational motion. Furthermore the case where rotational motion is only available around the gravity axis does not exhibit any rank loss anymore.

Rotation	RL
$w = 0$	2
$w \perp Cg$	1
$w \not\perp Cg$	0

TABLE IX
RANK DEFICIENCY FOR AN ARBITRARY NUMBER OF FLAT CONTACT.

References

[1] M Bloesch, C Gehring, P Fankhauser, M Hutter, M A Hoepflinger, and R Siegwart. State Estimation for Legged Robots on Unstable and Slippery Terrain. In *Intelligent Robots and Systems, IEEE/RSJ International Conference on*, 2013.