EKF Derivations

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Quaternion Review and Conventions

It is important to note that there are two different conventions for using quaternions, each of which is self-consistent; unfortunately, the quaternion algebra used in these conventions is often mixed up in the literature, resulting in inconsistent implementations [?]. We use that proposed as a standard by JPL [?] which was also used in [?] on which this work and that in [?] was based. A review of quaternions using this convention is given in this section.

Briefly, quaternions are one of several choices for representing $SO(3)$, the Lie group of rotations. The advantage of quaternions over other parameterizations is their numerical properties, efficiency and lack of singularities. We denote a quaternion by

$$q = \begin{pmatrix} q_x \\ q_y \\ q_z \\ q_w \end{pmatrix}$$

where q_w and $q_v = [q_x, q_y, q_z]^T$ are the scalar and vector components, respectively. Note that the ordering of these components does not depend on any conventions and is purely a matter of preference. We choose this ordering here to match the literature but actually implement quaternions in the reverse order to match the SL simulation environment.

All rotations are active, meaning that they act to rotate vectors; the quaternion representing the base orientation is written as $q = q_W^B$ which specifies a rotation from the world frame W to the base frame B. This quaternion corresponds to the rotation matrix $C = C[q]$ which rotates vectors defined in the world frame into the base frame.

Successive rotations about local axes are composed via left-multiplication, ie $R_A^C = R_B^C R_A^B$ represents a rotation from frame A to frame B (given in terms of the frame A basis) followed by a rotation from frame B to frame C (given in terms of the frame B basis). Analogously, we have $q_A^C = q_B^C \otimes q_A^B$ for quaternions where $\otimes$ denotes quaternion multiplication.

A vector in frame A is rotated into frame C as $v^C = R_A^C v^A$, with the reverse transformation given by $v^A = (R_A^C)^{-1} v^C$ where $(R_A^C)^{-1} = R_C^A = (R_A^C)^T$ since this is a rotation matrix (orthogonal matrix with det = +1). Vector rotations using quaternions are achieved through conjugation as

$$v^C = q_A^C \otimes \begin{pmatrix} v^A \\ 0 \end{pmatrix} \otimes (q_A^C)^{-1}$$

where $[v^A, 0]^T$ is called a pure quaternion and $(q_A^C)^{-1} = q_C^A = [-q_x, -q_y, -q_z, q_w]$ is the inverse or conjugate quaternion satisfying $q_A^C \otimes (q_A^C)^{-1} = (q_A^C)^{-1} \otimes q_A^C = q_I$ where $q_I = [0, 0, 0, 1]$ is the identity quaternion.

Quaternion multiplication is defined by

$$q \otimes p = \begin{pmatrix} q_w p_x + q_z p_y - q_y p_z + q_x p_w \\ -q_z p_x + q_w p_y + q_x p_z + qy p_w \\ q_y p_x - q_x p_y + q_w p_z + q_z p_w \\ -q_x p_x - q_y p_y - q_z p_z + q_w p_w \end{pmatrix}$$

This can be written more concisely as the matrix vector multiplication

$$\begin{aligned} q \otimes p = L(q)p &= \begin{pmatrix} q_w I - q_v^\times & q_v \\ -q_v^T & q_w \end{pmatrix} \begin{pmatrix} p_v \\ p_w \end{pmatrix} \\ &= R(p)q = \begin{pmatrix} p_w I + p_v^\times & p_v \\ -p_v^T & p_w \end{pmatrix} \begin{pmatrix} q_v \\ q_w \end{pmatrix} \end{aligned}$$

where $q_v = [q_x, q_y, q_z]^T$ is the vector part of q and

$$q_v^\times = \begin{pmatrix} 0 & -q_z & q_y \\ q_z & 0 & -q_x \\ -q_y & q_x & 0 \end{pmatrix}$$

is the skew-symmetric matrix corresponding to the vector q_v.

The rotation matrix corresponding the quaternion q is given by

$$\begin{aligned} C[q] &= (2q_w^2 - 1)I - 2q_w q_v^\times + 2q_v q_v^T \\ &= \begin{pmatrix} 2q_x^2 + 2q_w^2 - 1 & 2(q_x q_y + q_z q_w) & 2(q_x q_z - q_y q_w) \\ 2(q_x q_y - q_z q_w) & 2q_y^2 + 2q_w^2 - 1 & 2(q_y q_z + q_x q_w) \\ 2(q_x q_z + q_y q_w) & 2(q_y q_z - q_x q_w) & 2q_z^2 + 2q_w^2 - 1 \end{pmatrix} \end{aligned}$$

where the first equation can be seen as equivalent to Rodrigues' identity using the quaternion exponential map

$$\exp(\omega) = \begin{pmatrix} \sin(\frac{||\omega||}{2})\frac{\omega}{||\omega||} \\ \cos(\frac{||\omega||}{2}) \end{pmatrix}$$

which represents a rotation of $||\omega||$ about an axis $\omega/||\omega||$ as a quaternion. Letting $\delta\phi$ be an infinitesimal rotation, we see that

$$\delta q = \exp(\delta\phi) \approx \begin{pmatrix} \frac{1}{2}\delta\phi \\ 1 \end{pmatrix}$$

is the first-order approximation of an incremental quaternion. It follows from the definition of $C[q]$ that we have the first-order approximation

$$C[\delta q] \approx I - \delta\phi^{\times}$$

This can also be seen as the first-order approximation of the exponential map for rotation matrices

$$\exp(\delta\phi^{\times}) = \sum_{i=0}^{\infty} \frac{(-\delta\phi^{\times})^i}{i!} \approx I - \delta\phi^{\times}$$

It follows that the first-order expansion of q about a nominal quaternion $\bar{q}$ can be written in matrix form as

$$C[\delta q \otimes \bar{q}] = C[\delta q]C[\bar{q}] = (I - \delta\phi^{\times})\bar{C}$$

where $\bar{C} = C[\bar{q}]$. This approximation will be used in the derivations of the linearized filter dynamics in the next section.

The derivative of a quaternion is related to the angular velocity ω by the equation

$$\dot{q} = \frac{1}{2}\begin{pmatrix} \omega \\ 0 \end{pmatrix} \otimes q$$

and the first-order approximation of $\dot{\delta q}$ is given by

$$\dot{\delta q} \approx \begin{pmatrix} \frac{1}{2}\dot{\delta\phi} \\ 0 \end{pmatrix}$$

Linearization

Linearization of both the process and measurement models is performed analytically by expanding the filter states about their expected values using first-order Taylor series approximations. Products of small deviations are considered to be negligible, ultimately resulting in linear equations in terms of state deviations.

Process Model:

Position:

The original process model for the time-evolution of the position is

$$\dot{r} = v$$

Letting $r \approx \bar{r} + \delta r$ and $v \approx \bar{v} + \delta v$ leads to

$$\frac{d}{dt}(r + \delta r) = \dot{r} = \bar{v} + \delta v$$

From which it follows that

$$\dot{\bar{r}} + \dot{\delta r} = \bar{v} + \delta v$$

However, we know that the expected value of $\dot{r}$ is $\dot{\bar{r}} = \bar{v}$, so we finally have

$$\dot{\delta r} = \delta v$$

Velocity:

The original process model for the time-evolution of the velocity is

$$\dot{v} = C^T(\tilde{f} - b_f - w_f) + g$$

Let $C \approx (I - \delta\phi^\times)\bar{C}$, $v \approx \bar{v} + \delta v$, and $b_f \approx \bar{b}_f + \delta b_f$ so that

$$\frac{d}{dt}(\bar{v} + \delta v) = \dot{v} = \bar{C}^T(I + \delta\phi^\times)(\tilde{f} - (\bar{b}_f + \delta b_f) - w_f) + g$$

Simplifying yields

$$\dot{\bar{v}} + \dot{\delta v} = \bar{C}^T(I + \delta\phi^\times)(\bar{f} + \delta f) + g$$

where we have defined $\bar{f} = \tilde{f} - \bar{b}_f$ and $\delta f = -\delta b_f - w_f$ to be the "large-signal" and "small-signal" accelerations as in [**?**]. Expanding the right hand side yields

$$\dot{\bar{v}} + \dot{\delta v} = \bar{C}^T\bar{f} + \bar{C}^T\delta f + \bar{C}^T\delta\phi^\times\bar{f} + \bar{C}^T\delta\phi^\times\delta f + g$$

Recognizing that $\dot{\bar{v}} = \bar{C}^T\bar{f} + g$ is the expected value of the velocity process model equation, we are left with

$$\dot{\delta v} = \bar{C}^T\delta f + \bar{C}^T\delta\phi^\times\bar{f} + \bar{C}^T\delta\phi^\times\delta f$$

The last term on the right hand side is the (cross) product of two small vectors and can thus be neglected. This leaves

$$\dot{\delta v} = \bar{C}^T\delta f + \bar{C}^T\delta\phi^\times\bar{f}$$

Using the fact that $a^\times b = -b^\times a$ and expanding the expression for δf finally results in

$$\dot{\delta v} = -\bar{C}^T\bar{f}^\times\delta\phi - \bar{C}^T\delta b_f - \bar{C}^T w_f$$

Quaternion:

The original process model for the time-evolution of the quaternion is

$$\dot{q} = \frac{1}{2}\begin{pmatrix}\tilde{\omega} - b_\omega - w_\omega \\ 0\end{pmatrix} \otimes q$$

We define a deviation δq from the expected value $\bar{q}$ of the pose q by $q = \delta q \otimes \bar{q}$ and let $b_\omega \approx \bar{b}_\omega + \delta b_\omega$ so that

$$\begin{aligned} \dot{q} &= \frac{d}{dt}(\delta q \otimes \bar{q}) \\ \frac{1}{2}\begin{pmatrix}\tilde{\omega} - (\bar{b}_\omega + \delta b_\omega) - w_\omega \\ 0\end{pmatrix} \otimes q &= \dot{\delta q} \otimes \bar{q} + \delta q \otimes \dot{\bar{q}} \end{aligned}$$

Simplifying and using the fact that $\dot{\bar{q}} = \frac{1}{2}\begin{pmatrix}\tilde{\omega} - \bar{b}_\omega \\ 0\end{pmatrix} \otimes \bar{q}$ yields

$$\frac{1}{2}\begin{pmatrix}\tilde{\omega} - (\bar{b}_\omega + \delta b_\omega) - w_\omega \\ 0\end{pmatrix} \otimes q = \dot{\delta q} \otimes \bar{q} + \delta q \otimes \left(\frac{1}{2}\begin{pmatrix}\tilde{\omega} - \bar{b}_\omega \\ 0\end{pmatrix} \otimes \bar{q}\right)$$

Multiplying on the right of both sides by $\bar{q}^{-1}$ and recognizing that $\delta q = q \otimes \bar{q}^{-1}$ yields

$$\frac{1}{2}\begin{pmatrix}\tilde{\omega} - (\bar{b}_\omega + \delta b_\omega) - w_\omega \\ 0\end{pmatrix} \otimes \delta q = \dot{\delta q} + \delta q \otimes \frac{1}{2}\begin{pmatrix}\tilde{\omega} - \bar{b}_\omega \\ 0\end{pmatrix}$$

Solving for $\dot{\delta q}$ yields

$$\begin{aligned} \dot{\delta q} &= \frac{1}{2}\begin{pmatrix}\tilde{\omega} - (\bar{b}_\omega + \delta b_\omega) - w_\omega \\ 0\end{pmatrix} \otimes \delta q - \delta q \otimes \frac{1}{2}\begin{pmatrix}\tilde{\omega} - \bar{b}_\omega \\ 0\end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix}\tilde{\omega} - \bar{b}_\omega \\ 0\end{pmatrix} \otimes \delta q - \delta q \otimes \frac{1}{2}\begin{pmatrix}\tilde{\omega} - \bar{b}_\omega \\ 0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}-\delta b_\omega - w_\omega \\ 0\end{pmatrix} \otimes \delta q \end{aligned}$$

where the second step results from the fact that we can split a pure quaternion into the sum of multiple pure quaternions (since they are just vectors) and distribute them over quaternion multiplication. For the quaternion δq corresponding to the small rotation $\delta\phi$, we may write $\delta q \approx \begin{pmatrix}\frac{1}{2}\delta\phi \\ 1\end{pmatrix}$ and thus $\dot{\delta q} \approx \begin{pmatrix}\frac{1}{2}\dot{\delta\phi} \\ 0\end{pmatrix}$ so that

$$\begin{pmatrix} \frac{1}{2}\dot{\delta\phi} \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \tilde{\omega} - \bar{b}_\omega \\ 0 \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2}\delta\phi \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2}\delta\phi \\ 1 \end{pmatrix} \otimes \frac{1}{2}\begin{pmatrix} \tilde{\omega} - \bar{b}_\omega \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -\delta b_\omega - w_\omega \\ 0 \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2}\delta\phi \\ 1 \end{pmatrix}$$

Using the fact that quaternion products can be written as matrix-vector products as shown in the previous section and letting $\hat{\omega} = \tilde{\omega} - \bar{b}_\omega$, we can write the above as

$$\begin{aligned}
\begin{pmatrix} \frac{1}{2}\dot{\delta\phi} \\ 0 \end{pmatrix} &= \frac{1}{2}\left[\begin{pmatrix} -\hat{\omega}^\times & \hat{\omega} \\ -\hat{\omega}^T & 0 \end{pmatrix} - \begin{pmatrix} \hat{\omega}^\times & \hat{\omega} \\ -\hat{\omega}^T & 0 \end{pmatrix}\right]\begin{pmatrix} \frac{1}{2}\delta\phi \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -(-\delta b_\omega - w_\omega)^\times & (-\delta b_\omega - w_\omega) \\ -(-\delta b_\omega - w_\omega)^T & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2}\delta\phi \\ 1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} -2\hat{\omega}^\times & 0 \\ 0^T & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2}\delta\phi \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -\delta b_\omega - w_\omega \\ 0 \end{pmatrix}
\end{aligned}$$

where the second step results from simplifying and eliminating products of small quantities (products of deviations and products of noise with deviations) in the second term. Multiplying out this expression yields the equations

$$\begin{aligned}
\frac{1}{2}\dot{\delta\phi} &= -\frac{1}{2}\omega^\times\delta\phi - \frac{1}{2}(\delta b_\omega + w_\omega) \\
0 &= 0
\end{aligned}$$

where the first equation results in the linearized orientation dynamics

$$\dot{\delta\phi} = -\omega^\times\delta\phi - \delta b_\omega - w_\omega$$

and the second equation ensures consistency.

Foot Position:

The original process model for the time-evolution of the position of the i^{th} foot is

$$\dot{p} = C^T w_p$$

Again, let $C = (I - \delta\phi^\times)\bar{C}$ so that we have $C^T \approx \bar{C}^T(I + \delta\phi^\times)$. Also let $p \approx \bar{p} + \delta p$ so that

$$\frac{d}{dt}(\bar{p} + \delta p) = \dot{p} = \bar{C}^T(I + \delta\phi^\times)w_p$$

Simplifying the above yields

$$\dot{\bar{p}} + \dot{\delta p} = \bar{C}^T w_p + \bar{C}^T \delta\phi^\times w_p$$

The second term is the cross product of a state deviation and a (small) noise vector and thus is neglected. Further, we know that

$$\dot{\bar{p}} = E[\dot{p}] = E[C^T w_p] = E[w_p] = 0$$

since w_p is zero-mean and since the rotation matrix C does not alter the statistics of w_p. It directly follows that

$$\dot{\delta p} = \bar{C}^T w_p$$

Accelerometer/Gyroscope Bias:

The original process model for the time-evolution of the accelerometer bias is

$$\dot{b}_f = w_{b_f}$$

Letting $b_f \approx \bar{b}_f + \delta b_f$ yields

$$\frac{d}{dt}(\bar{b}_f + \delta b_f) = \dot{b}_f = w_{b_f}$$

It follows that

$$\dot{\bar{b}}_f + \delta\dot{b}_f = w_{b_f}$$

However, we know that $\dot{\bar{b}}_f = E[\dot{b}_f] = E[w_{b_f}] = 0$ since w_{b_f} is zero mean and thus

$$\delta\dot{b}_f = w_{b_f}$$

Likewise, for the gyroscope bias we have

$$\delta\dot{b}_\omega = w_{b_\omega}$$

Foot Quaternion:

The continuous process model for the time-evolution of the foot quaternion is

$$\dot{z} = \frac{1}{2}\begin{pmatrix} w_z \\ 0 \end{pmatrix} \otimes z$$

Let $z \approx \delta z \otimes \bar{z}$ so that $\dot{z} = \delta\dot{z} \otimes \bar{z} + \delta z \otimes \dot{\bar{z}}$. Then we have

$$\delta\dot{z} \otimes \bar{z} + \delta z \otimes \dot{\bar{z}} = \frac{1}{2}\begin{pmatrix} w_z \\ 0 \end{pmatrix} \otimes z$$

However, $\dot{\bar{z}} = 0$ is the expected value of $\dot{z}$ since $E[w_z] = 0$. We are thus left with

$$\delta\dot{z} \otimes \bar{z} = \frac{1}{2}\begin{pmatrix} w_z \\ 0 \end{pmatrix} \otimes z$$

Multiplying both sides on the left by $\bar{z}^{-1}$ and using the fact that $\delta z = z \otimes \bar{z}^{-1}$, we have

$$\delta\dot{z} = \frac{1}{2}\begin{pmatrix} w_z \\ 0 \end{pmatrix} \otimes \delta z$$

Again, writing the quaternion product as a matrix-vector product leads to

$$\begin{pmatrix} \frac{1}{2}\delta\dot{\theta} \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -w_z^{\times} & w_z \\ w_z^T & 0 \end{pmatrix}\begin{pmatrix} \delta\theta \\ 0 \end{pmatrix}$$

Multiplying out the above yields

$$\begin{pmatrix} \frac{1}{2}\delta\dot{\theta} \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4}w_z^{\times}\delta\theta + \frac{1}{2}w_z \\ \frac{1}{4}w_z^T\delta\theta \end{pmatrix}$$

After eliminating products of small quantities, the first equation yields

$$\delta\dot{\theta} = w_z$$

as desired, and the second yields $0 = 0$ ensuring consistency.

Measurement Model:

Relative Foot Position:

The relative foot position measurement is given by (for a single foot)

$$s_p = C(p - r) + n_p$$

Letting $s_p \approx \bar{s}_p + \delta s_p$, $C \approx (I - \delta\phi^\times)\bar{C}$, $p \approx \bar{p} + \delta p$ and $r \approx \bar{r} + \delta r$ we have

$$\bar{s}_p + \delta s_p = s_p = (I - \delta\phi^\times)\bar{C}((\bar{p} + \delta p) - (\bar{r} + \delta r)) + n_p$$

Expanding the above yields

$$\bar{s}_p + \delta s_p = \bar{C}(\bar{p} - \bar{r}) + \bar{C}(\delta p - \delta r) - \delta\phi^\times \left(\bar{C}(\bar{p} - \bar{r})\right) - \delta\phi^\times \left(\bar{C}(\delta p - \delta r)\right) + n_p$$

Recognizing that $\bar{s}_p = \bar{C}(\bar{p} - \bar{r}) + n_p$ this simplifies to

$$\delta s_p = \bar{C}(\delta p - \delta r) - \delta\phi^\times \left(\bar{C}(\bar{p} - \bar{r})\right) - \delta\phi^\times \left(\bar{C}(\delta p - \delta r)\right)$$

The last term above is the cross product of state deviations and is thus neglected. It follows that, after using the fact that $a^\times b = -b^\times a$, we have

$$\delta s_p = -\bar{C}\delta r + \bar{C}\delta p + \left(\bar{C}(\bar{p} - \bar{r})\right)^\times \delta\phi$$

Relative Foot Quaternion:

The relative foot quaternion measurement is given by (for a single foot)

$$s_z = n_z \otimes q \otimes z^{-1}$$

Letting $s_z \approx \delta s_z \otimes \bar{s}_z$, $q \approx \delta q \otimes \bar{q}$ and $z \approx \delta z \otimes \bar{z}$ we have

$$\delta s_z \otimes \bar{s}_z = s_z = n_z \otimes (\delta q \otimes \bar{q}) \otimes (\delta z \otimes \bar{z})^{-1}$$

After expanding the right hand side and regrouping, we have

$$\delta s_z \otimes \bar{s}_z = n_z \otimes \delta q \otimes (\bar{q} \otimes \bar{z}^{-1}) \otimes \delta z^{-1}$$

Substituting $\bar{s}_z = \bar{q} \otimes \bar{z}^{-1}$ in the right hand side and multiplying on the right of both sides by $\bar{s}_z^{-1}$ yields

$$\delta s_z = n_z \otimes \delta q \otimes \left(\bar{s}_z \otimes \delta z^{-1} \otimes \bar{s}_z^{-1}\right)$$

In [**?**] was shown that a triple product of quaternions can be written as

$$\left(q \otimes p \otimes q^{-1}\right) = \begin{pmatrix} C[q]p_v \\ p_s \end{pmatrix}$$

Since δz^{-1} is the quaternion corresponding to the small rotation $-\delta\theta$, we have the approximation

$$\delta z^{-1} \approx \begin{pmatrix} -\frac{1}{2}\delta\theta \\ 1 \end{pmatrix}$$

and thus

$$\left(\bar{s}_z \otimes \delta z^{-1} \otimes \bar{s}_z^{-1}\right) \approx \begin{pmatrix} -\frac{1}{2}C[\bar{s}_z]\delta\theta \\ 1 \end{pmatrix}$$

Assuming that the rotations corresponding to δs_z and n_z are small and rewriting the quaternion products as matrix-vector products yields

$$\begin{pmatrix} \frac{1}{2}\delta s_z \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}n_z \\ 1 \end{pmatrix} \otimes \left[\begin{pmatrix} I - \frac{1}{2}\delta\phi^\times & \frac{1}{2}\delta\phi \\ -\frac{1}{2}\delta\phi^T & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2}C[\bar{s}_z]\delta\theta \\ 1 \end{pmatrix} \right]$$

$$= \begin{pmatrix} \frac{1}{2}n_z \\ 1 \end{pmatrix} \otimes \begin{pmatrix} -\frac{1}{2}C[\bar{s}_z]\delta\theta + \frac{1}{4}\delta\phi^\times C[\bar{s}_z]\delta\theta + \frac{1}{2}\delta\phi \\ \frac{1}{4}\delta\phi^T C[\bar{s}_z]\delta\theta + 1 \end{pmatrix}$$

$$= \begin{pmatrix} I - \frac{1}{2}n_z^\times & \frac{1}{2}n_z \\ -\frac{1}{2}n_z^T & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2}C[\bar{s}_z]\delta\theta + \frac{1}{4}\delta\phi^\times C[\bar{s}_z]\delta\theta + \frac{1}{2}\delta\phi \\ \frac{1}{4}\delta\phi^T C[\bar{s}_z]\delta\theta + 1 \end{pmatrix}$$

where δs_z is the vector corresponding to the measurement quaternion deviation, n_z is the measurement noise vector and $\delta\phi$ is the vector corresponding to the deviation in the base pose. The expression on the right can be simplified by eliminating terms involving products of small quantities. This yields

$$\begin{pmatrix} \frac{1}{2}\delta s_z \\ 1 \end{pmatrix} = \begin{pmatrix} I - \frac{1}{2}n_z^\times & \frac{1}{2}n_z \\ -\frac{1}{2}n_z^T & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2}C[\bar{s}_z]\delta\theta + \frac{1}{2}\delta\phi \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2}C[\bar{s}_z]\delta\theta + \frac{1}{2}\delta\phi + \frac{1}{4}n_z^\times C[\bar{s}_z]\delta\theta - \frac{1}{4}n_z^\times \delta\phi + \frac{1}{2}n_z \\ \\ \frac{1}{4}n_z^T C[\bar{s}_z]\delta\theta - \frac{1}{4}n_z^T \delta\phi + 1 \end{pmatrix}$$

After again eliminating terms involving products of small quantities and simplifying (using the fact that $C^T[q] = C[q^{-1}]$), the first equation yields the linearized measurement

$$\delta s_z = -C[\bar{q} \otimes \bar{z}^{-1}]\delta\theta + \delta\phi + n_z$$

The second equation becomes $1 = 1$, ensuring consistency.